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DIRECT PRODUCTS AND THE INTERSECTION MAP OF CERTAIN CLASSES OF FINITE GROUPS

Julia Chifman
University of Kentucky, julia.chifman@gmail.com

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Dr. James C. Beidleman, Major Professor

, Director of Graduate Studies

ABSTRACT OF DISSERTATION

Julia Chifman

The Graduate School
University of Kentucky
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ABSTRACT OF DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Julia Chifman
Lexington, Kentucky

Director: Dr. James C. Beidleman, Professor of Mathematics
Lexington, Kentucky 2009

ABSTRACT OF DISSERTATION

DIRECT PRODUCTS AND THE INTERSECTION MAP OF CERTAIN CLASSES OF FINITE GROUPS.

The main goal of this work is to examine classes of finite groups in which normality, permutability and Sylow-permutability are transitive relations. These classes of groups are called $\mathcal{T}$, $\mathcal{PT}$ and $\mathcal{PST}$, respectively. The main focus is on direct products of $\mathcal{T}$, $\mathcal{PT}$ and $\mathcal{PST}$ groups and the behavior of a collection of *cyclic* normal, permutable and Sylow-permutable subgroups under the intersection map.

In general, a direct product of finitely many groups from one of these classes does not belong to the same class, unless the orders of the direct factors are relatively prime. Examples suggest that for solvable groups it is not required to have relatively prime orders to stay in the class. In addition, the concept of normal, permutable and S-permutable cyclic sensitivity is tied with that of $\mathcal{T}_c$, $\mathcal{PT}_c$ and $\mathcal{PST}_c$ groups, in which *cyclic* subnormal subgroups are normal, permutable or Sylow-permutable. In the process another way of looking at the Dedekind, Iwasawa and nilpotent groups is provided as well as possible interplay between direct products and the intersection map is observed.

KEYWORDS: S-permutable, permutable, direct product, intersection map, cyclic subgroups.

Author's signature: Julia Chifman

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By
Julia Chifman

Director of Dissertation: James C. Beidleman

Director of Graduate Studies: Qiang Ye

Date: March 23, 2009

RULES FOR THE USE OF DISSERTATIONS

Name	Date

DISSERTATION

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Dedicated to my husband, Igor Chifman. Without his unconditional love and support this work would not be possible.

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Overview

Groups are the main components of algebraic structures such as rings, fields, modules and vector spaces. They have many applications in chemistry, physics and last but not least, algebraic biology.

The main goal of this work is to examine classes of finite groups with certain transitivity properties, their direct products and the behavior of a collection of subgroups under the intersection map. A direct product is an important mathematical concept that is defined on already known objects, in this case groups, and it provides a way to build new groups as well as to analyze groups from its direct factors. It is not always true that given two groups from one class, the new group formed using a direct product will stay in the same class. Classes of groups with certain transitivity properties have been studied and characterized by many authors, however characterizations of the direct product with respect to these classes have not been provided. Therefore the task is to find necessary and sufficient conditions for a direct product to stay in the same class.

The intersection map of subgroups is another important concept that was tied by numerous authors to the classes of group with certain transitivity properties, but there are no known generalizations of their results in connection to the wider classes of groups introduced in recent years. The latter is the question of interest.

The following sections provide a brief overview of the Chapters 1, 2 and 3.

Certain classes of groups

Over the past fifty years finite solvable groups in which normality, permutability and Sylow-permutability are transitive relations have been studied by many authors with Gaschütz [8], Zacher [17] and Agrawal [1] being the first pioneers. As students, when we take a modern algebra course, we quickly learn that if H is a normal subgroup of K and K is a normal subgroup of the group G, then it is not necessarily true that H will be a normal subgroup of G. Groups in which normality is a transitive relation are called $\mathcal{T}$-groups. In other words, $\mathcal{T}$-groups are precisely the groups in which every subnormal subgroup is normal. Gaschütz [8] in 1957 provided a characterization of solvable $\mathcal{T}$-groups.

Zacher [17] and Agrawal [1] went on further by asking a similar question about permutability and Sylow-permutability being transitive relations, respectively. A subgroup H of the group G is called *permutable* (*Sylow-permutable*) if $HP = PH$ for all subgroups (Sylow subgroups) P of G. $\mathcal{PT}$ will denote the class of groups in which permutability is a transitive relation and $\mathcal{PST}$ will denote the class of groups in which Sylow-permutability is a transitive relation. Ore [11] proved that permutable subgroups are subnormal and Kegel [10] has proved that Sylow-permutable subgroups are subnormal as well. A direct consequence of the latter is that $\mathcal{PT}$ ($\mathcal{PST}$) groups are precisely the groups in which every subnormal subgroup is permutable (Sylow-permutable). Since normal subgroups are permutable and permutable subgroups are Sylow-permutable then it follows that $\mathcal{T} \subset \mathcal{PT} \subset \mathcal{PST}$. Also, note that the containment is proper, since a dihedral group of order 8 is a $\mathcal{PST}$-group but not a $\mathcal{PT}$-group and the non-Dedekind modular group of order 16 is a $\mathcal{PT}$-group but not a $\mathcal{T}$-group. Zacher [17] in 1964 and Agrawal [1] in 1975 have provided characterizations for solvable $\mathcal{PT}$ and $\mathcal{PST}$ groups, respectively.

Direct Products of Certain Classes of Finite Groups

Direct products of finite solvable groups in which normality, permutability and Sylow-permutability are transitive relations was the first topic that I have addressed. Some ideas and results presented in Chapter 2 were accepted for publication in *Communications in Algebra*, [7].

When we study, for example, nilpotent groups we ask the question whether a direct product of finitely many nilpotent groups is nilpotent. A direct product of known objects gives us a new object, and it is preferable that our new object has similar attributes as its direct factors.

It was natural to ask the question about direct products of $\mathcal{T}$, $\mathcal{PT}$ and $\mathcal{PST}$-groups. Part of the answer is found in [1]; *Agrawal has shown that if G_1 and G_2 are $\mathcal{PST}$ groups such that their orders are relatively prime, then $G_1 \times G_2$ is a $\mathcal{PST}$-group.* However, it is not necessary for solvable groups to have relatively prime orders for the product to remain in the class. For example, consider a dihedral group of order 12, which is a direct product of a symmetric group of order 6, S_3, and a cyclic group of order 2, C_2. Notice that the orders of S_3 and C_2 are not relatively prime and yet $S_3 \times C_2$ is a solvable $\mathcal{PST}$-group, while the group $S_3 \times C_3$, where C_3 is a cyclic group of order three, is not a $\mathcal{PST}$-group.

It turns out that the key lies in the nilpotent residual of the group. In the following theorem, L_i stands for the nilpotent residual of the group G_i, by which we mean $L_i = \bigcap\{H_i | H_i \lhd G_i \text{ and } G_i/H_i \text{ is nilpotent }\}$.

Theorem. (2.1.2) *Let G_1 and G_2 be finite groups. $G_1 \times G_2$ is a solvable $\mathcal{PST}$-group if and only if G_1 and G_2 are solvable $\mathcal{PST}$ groups and $(|L_i|, |G_j|) = 1$ for $i \neq j$ and $i, j \in \{1, 2\}$.*

Theorem 2.1.2 is true for a direct product of finitely many solvable $\mathcal{PST}$-groups. To extend Theorem 2.1.2 to the classes of solvable $\mathcal{T}$ and $\mathcal{PT}$ groups the focus had to be shifted to their Sylow subgroups. If G_1 and G_2 are solvable $\mathcal{PT}$ ($\mathcal{T}$) groups and $(|L_i|, |G_j|) = 1$ for $i \neq j$, then Theorem 2.1.2 implies that $G_1 \times G_2$ is a solvable $\mathcal{PST}$-group. Agrawal [1] has shown that a $\mathcal{PST}$ group G is a $\mathcal{PT}$ ($\mathcal{T}$) group if all of its Sylow subgroups are *Iwasawa* (*Dedekind*) respectively. Thus, the question was: *When is the direct product of two or more Iwasawa (Dedekind) p-groups will be an Iwasawa (Dedekind) group?* By an *Iwasawa* (*Dedekind*) group we mean a group in which every subgroup is permutable (normal).
It follows directly from the Dedekind-Baer Theorem [13] when the direct product of two Dedekind p-groups is again a Dedekind group. In the case of Iwasawa p-groups, there is a nice result in [15] that gives a precise description of such p-groups, which was first stated in 1941 by Iwasawa himself. But, if P_1 and P_2 are two Iwasawa p-groups for the same prime p and Q_8, the Quaternion group of order 8, is not a subgroup of P_1 nor P_2 in case $p = 2$, then it was not clear when a direct product $P_1 \times P_2$ will be again an Iwasawa group. For example, let G be a modular group of order 16 with presentation $G = \langle x, y | x^8 = y^2 = 1, x^y = x^5 \rangle$ and C_2 and C_8 be cyclic groups of order 2 and 8, respectively. Note, G is a non-abelian Iwasawa group. The direct product $G \times C_2$ is an Iwasawa group but $G \times C_8$ is not.
A direct consequence of Iwasawa's theorem [15] is that if P_i is non-abelian Iwasawa p-group and Q_8 is not contained in P_i, then $P_i = A_i \langle x_i \rangle$ where A_i is abelian normal subgroup of P_i and P_i/A_i is cyclic; furthermore there is a positive integer s_i such that $a^{x_i} = a^{1+p^{s_i}}$ for all $a \in A_i$ with $s_i \geq 2$ in case $p = 2$.
Using the latter, I was able to show that if $P := P_1 \times \cdots \times P_n$ is non-abelian Iwasawa p-group so that $Q_8 \not\leq P$, then *exactly* one P_i will be non abelian Iwasawa p-group and $P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n$ must be an abelian p-group such that the *exponent* of $P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n$ is less than or equal to p^{s_i}, where s_i is the positive integer as described above. By the *exponent* of the group G, denoted by $\mathrm{Exp}(G)$, we mean the smallest positive integer n such that $x^n = 1$ for all $x \in G$. The next theorem will employ the following notation: let $P := P_1 \times \cdots \times P_n$ for $n \geq 2$ then $P \backslash P_i := P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n$.

Theorem. (2.3.2) *Let $G := G_1 \times \cdots \times G_n$ be a finite group and $P := P_1 \times \cdots \times P_n \in \mathrm{Syl}_p(G)$ where $P_i \in \mathrm{Syl}_p(G_i)$ for some prime p dividing the order of G and $n \geq 2$. Then G is a solvable $\mathcal{PT}$-group if and only if the following hold:*

(i) $G_1, \ldots, G_n$ are solvable $\mathcal{PT}$-groups such that $(|\gamma_*(G_i)|, |G_j|) = 1$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

(ii) If Q_8 is contained in P, then it is a subgroup of exactly one P_i and $P \backslash P_i$ is elementary 2-abelian or trivial.

(iii) If Q_8 is not contained in P, in case $p = 2$, and P is not abelian, then exactly one P_i is non-abelian and $P \backslash P_i$ is an abelian p-group such that $\mathrm{Exp}(P \backslash P_i) \leq p^{s_i}$.

A similar result to that of Theorem 2.3.2 can be obtained for direct products of solvable $\mathcal{T}$-groups with the difference that if Q_8 is not contained in P then P is an abelian group.
In 2005 D.Robinson in [12] introduced three new classes of groups. He called them $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$. A group G is a $\mathcal{PST}_c$-group if every cyclic subnormal subgroup of G is Sylow-permutable in G. Similarly, classes $\mathcal{PT}_c$ and $\mathcal{T}_c$ are defined by requiring cyclic subnormal subgroups to be permutable or normal, respectively. Robinson in [12] provided characterizations for both solvable and insolvable cases. I have extended some of the above results to solvable $\mathcal{PST}_c$, $\mathcal{PT}_c$, and $\mathcal{T}_c$ groups.

The Intersection Map of Subgroups

The intersection map of subgroups in connection to the classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$ is a collaborative work with my advisor James Beidleman. Parts of this work were published in Ricerche di Matematica [5].
At that time my advisor and Matthew Ragland were locally analyzing the intersection map in connection with $\mathcal{T}$, $\mathcal{PT}$ and $\mathcal{PST}$ groups. They have generalized in [6] Theorem 1 of Bauman [4] to $\mathcal{PT}$ and $\mathcal{PST}$ groups. Beidleman asked if similar generalizations are possible for the classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$.
A subgroup H of the group G is said to be *normal sensitive* if whenever X is a normal subgroup of H there is a normal subgroup Y of G such that $X = Y \cap H$, that is if the map, known as the intersection map, $Y \mapsto H \cap Y$ sends the lattice of normal subgroups of G onto the lattice of normal subgroups of H. *Permutable sensitive* (*S-permutable sensitive*) are defined in the similar fashion by requiring X and Y to be permutable (Sylow-permutable) subgroups of H and G respectively. Note, that the

set of all permutable subgroups need not be a sublattice of the lattice of subnormal subgroups of a group G. Hence, in the case of permutability the intersection map is not necessarily a lattice map. An example that addresses the latter can be found in [6] on page 220, Example 1. On the other hand, the collection of S-permutable subgroups is a sublattice of the lattice of subnormal subgroups of G. For details the reader may consult [10] and [14].
Bauman [4], Beidleman and Ragland [6], have tied the concept of normal, permutable and S-permutable sensitivity with $\mathcal{T}$, $\mathcal{PT}$ and $\mathcal{PST}$ groups. They have showed that G is a solvable $\mathcal{T}$, $\mathcal{PT}$, or $\mathcal{PST}$ group if and on if every subgroup of G is normal, permutable, or S-permutable sensitive in G, respectively. In addition, Beidleman and Ragland in [6] went on further by asking a question whether one can restrict S-permutable, permutable, and normal sensitivity to normal subgroups and deduce that G is still a $\mathcal{PST}$, $\mathcal{PT}$, or a $\mathcal{T}$ group respectively. While they have affirmatively answered the question about $\mathcal{PST}$ and $\mathcal{T}$ groups in [6], the question about permutable sensitivity restricted to normal subgroups of *solvable* groups was answered later in [3]. Our interest was in developing similar connections with classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$. In particular, if we restrict the intersection map to *cyclic* subgroups, then what can we say about the behavior of a collection of *cyclic* normal, permutable and S-permutable subgroups under this restricted intersection map. The definition of the intersection map restricted to cyclic subgroups together with two Lemmas by R.Schmidt [15] and P.Schmid [14] is equivalent to the following definition.

Definition. *A subgroup H of the group G is normal (permutable or S-permutable) cyclic sensitive in G if every normal (permutable or Sylow-permutable) cyclic subgroup of H is normal (permutable or Sylow-permutable) subgroup of G.*

We have tied the concept of normal, permutable and S-permutable cyclic sensitivity with that of $\mathcal{T}_c$, $\mathcal{PT}_c$ and $\mathcal{PST}_c$ groups [5]. In the process we provided another way of looking at Dedekind, Iwasawa and nilpotent groups.

Theorem. (3.2.6) *Let G be a finite group.*

1. *G is a nilpotent group if and only if every subgroup of G is S-permutable cyclic sensitive.*

2. *G is an Iwasawa group if and only if every subgroup of G is permutable cyclic sensitive.*

3. *G is a Dedekind group if and only if every subgroup of G is normal cyclic sensitive.*

If we replace "every subgroup" in Theorem 3.2.6 by "every subnormal subgroup" then we get that G is a $\mathcal{PST}_c$,$\mathcal{PT}_c$, or a $\mathcal{T}_c$ group, respectively.
Robinson in [12] has proved that if every subgroup of a group G is $\mathcal{PST}_c$ then G is a solvable $\mathcal{PST}$ group. The same is true for solvable $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups. Robinson's results motivate the following theorem that relates cyclic sensitivity to solvable $\mathcal{PST}$, $\mathcal{PT}$ and $\mathcal{T}$ groups.

Theorem. (3.2.10) *Let G be a finite group.*

1. *G is a solvable $\mathcal{PST}$-group if and only if every subnormal subgroup of H is S-permutable cyclic sensitive in H for all subgroups H of G.*

2. *G is a solvable $\mathcal{PT}$-group if and only if every subnormal subgroup of H is permutable cyclic sensitive in H for all subgroups H of G.*

3. *G is a solvable $\mathcal{T}$-group if and only if every subnormal subgroup of H is normal cyclic sensitive in H for all subgroups H of G.*

Since cyclic subgroups can be written as a direct product of cyclic p-groups of relatively prime orders, it was natural to look at the Sylow subgroups. In addition, if X is any subnormal cyclic subgroup of a group G then X is contained in the Fitting subgroup of G and in particular the Sylow p-subgroup of X, for some prime dividing the order of X, lies in the Sylow p-subgroup of the Fitting subgroup. Details on the Sylow subgroups and the intersection map are in Chapter 3.

Notation

$G, H, \ldots$	Sets and groups
$g, h, \ldots$	Elements of a set
$\mathcal{G}, \mathcal{H} \ldots$	Classes of groups
$H \leq G$	H is a subgroup of G
$H < G$	H is a proper subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$H \lhd \lhd G$	H is a subnormal subgroup of G
$\mathrm{Syl}(G)$	Set of Sylow subgroups of G
$\mathrm{Syl}_p(G)$	Set of Sylow p-subgroups of G
HK	$\{ hk \mid h \in H, k \in K\}$
G/H	Factor group when $H \trianglelefteq G$
$\langle X \rangle$	Group generated by X
$\langle X, g \rangle$	$\langle X \cup \{g\} \rangle$
$\lvert H \rvert$	Order of H
$\mathrm{ord}(h)$	Order of h
$(G : H)$	Index of H in G
$Z(G)$	Center of G
$C_G(H)$	Centralizer of H in G
$N_G(H)$	Normalizer of H in G
$\mathrm{Aut}(G)$	Group of automorphisms of G
$H \times K$	Direct product of H with K
$H \rtimes K$	Semidirect product of H with K
$\mathrm{Fit}(G)$	the Fitting subgroup of G
$\gamma_*(G)$	Last term of the lower central series of G
$O^\pi(G)$	Group generated by all the π'-elements of G
$O_\pi(G)$	Product of all normal π-subgroups of G
g^h	$h^{-1}gh$
$[g, h]$	$g^{-1}h^{-1}gh$
$[H, K]$	Group generated by all $[h, k]$ with $h \in H$ and $k \in K$
$G' = [G, G]$	Derived subgroup of G
C_n	Cyclic group of order n
S_n	Symmetric group of order $n!$
D_{2n}	Dihedral group of order $2n$
Q_8	Quaternion group of order 8

Chapter 1 Fundamental concepts and certain classes of finite groups

This chapter provides an overview of the fundamental concepts necessary for the consequent chapters, with the underlying assumption that the reader has some familiarity with group theory. For further details on fundamentals of group theory and any unexplained notation the reader may consult [13], [15] and [16].

1.1 Sylow and Hall subgroups

Let G be a finite group and p a prime number dividing the order of G. G is called a *p-group* if the order of G is a power of p, that is $|G| = p^n$ for some positive integer n. Subgroups of G that are p-groups are called *p-subgroups.* A maximal p-subgroup of G is called a *Sylow p-subgroup* of G and we will denote the set of Sylow p-subgroups of G by $\mathrm{Syl}_p(G)$. The fundamental result of finite p-groups is that *a nontrivial finite p-group has a nontrivial center.*

Theorem 1.1.1. (1) (Cauchy's Theorem) *If G is a finite group and p is a prime dividing $|G|$ then G has an element of order p.*

(2) (Sylow's Theorem) *Let G be a finite group of order $p^\alpha m$, where p is a prime not dividing m.*

(i) *Sylow p-subgroups of G exist and they are all conjugate (that is isomorphic).*

(ii) *If n_p is the number of Sylow p-subgroups then $n_p \equiv 1 \mod p$ and $n_p|m$.*

The second part of (i) in Sylow's Theorem states that if $P \in \mathrm{Syl}_p(G)$ then for all $g \in G$, $P^g \in \mathrm{Syl}_p(G)$, that is if P_1 and P_2 are two Sylow p-subgroups of G then there exist an element g in G so that $P_1 = P_2^g$. In addition, Sylow's theorem implies that Sylow p-subgroup P is normal in G if and only if P is the unique Sylow p-subgroup. Now, let π be a set of primes, and $\pi' = \{primes \;\; p \mid p \notin \pi\}$. A positive integer n is called a *π-number* if all prime divisors of n belong to π. A subgroup H of a finite group G is called a *π-group* if the order of H is a π-number. Thus, a *Sylow π-subgroup* of G is defined to be a maximal π-subgroup. For finite groups Sylow π-subgroups exist but in general they are not conjugate.

P. Hall in [9] extended Sylow's theorem to solvable groups (solvable groups are defined shortly) where he proved the existence of a subgroup with order relatively prime to its index and showed that any two such subgroups are conjugate. This paper of P. Hall [9] and others that followed revolutionized the theory of finite solvable groups. Now we provide a general definition of what is known as a *Hall subgroup*.

Definition 1.1.2. H is called a *Hall-subgroup* of a group G if $(|H|, (G : H)) = 1$. In particular, a *Hall π-subgroup* of G is a subgroup H of G such that $|H|$ is a π-number and $(G : H)$ is a π' -number.

If $\pi = \{p\}$ for some prime p, then π-group and p-group mean the same, that is Hall p-subgroup is the same as a Sylow p-subgroup and every Hall π-subgroup is a Sylow π-subgroup.

Definition 1.1.3. Let G be a group and $N \lhd G$. A *complement* of N in G is a subgroup H of G such that $H \cap N = 1$ and $G = HN$.

The following theorem is a fundamental result of finite group theory.

Theorem 1.1.4 (Schur-Zassenhaus Theorem)**.** *Let G be a finite group and N a normal Hall π'-subgroup of G. Then N has a complement in G and all the complements of N in G are conjugate.*

1.2 Nilpotent, Solvable and Supersolvable groups

Let G be any group and consider the series for G

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G, \tag{1.1}$$

where G_i is a subgroup of G for all $i \in \{1, \ldots, n-1\}$. If each $G_i \lhd G$ then 1.1 is called a *normal series*. If $G_i \lhd G_{i+1}$ only then 1.1 is called a *subnormal series*.
We say that a subgroup H of G is *subnormal* in G if there exist a subnormal series between H and G.

Definition 1.2.1.

1. A group G is called *nilpotent* if it has a central series, that is a normal series such that $G_{i+1}/G_i \in Z(G/G_i)$ for all $i \in \{1, \ldots, n-1\}$.

2. A group G is called *solvable* if it has an abelian series, that is a subnormal series such that G_{i+1}/G_i is abelian, for all $i \in \{1, \ldots, n-1\}$.

3. A group G is called *supersolvable* if it has a cyclic series, that is a normal series such that G_{i+1}/G_i is cyclic, for all $i \in \{1, \ldots, n-1\}$.

Example 1.2.2. Consider the following groups:

$$D_8 = \langle x, y \mid x^4 = y^2 = 1,\ x^y = x^{-1} \rangle$$

$$S_3 = \langle x, y \mid x^3 = y^2 = 1,\ (xy)^2 = 1 \rangle$$

$$S_4 = \langle x, y \mid x^3 = y^2 = 1,\ (xy)^4 = 1 \rangle$$

1. *Nilpotent:* D_8

2. *Supersolvable:* D_8, S_3

3. *Solvable:* D_8, S_3, S_4

Note, S_3 is not nilpotent and S_4 is not supersolvable.

From the above definition 1.2.1 and the example 1.2.2 we see that the class of finite nilpotent groups, supersolvable groups and solvable groups are proper subclasses of each other, that is

$$\{\text{finite nilpotent groups}\} \subset \{\text{finite supersolvable groups}\} \subset \{\text{finite solvable groups}\}.$$

It is important to note that p-groups are nilpotent. Also, we would like to state some important characterizations of finite nilpotent groups.

Theorem 1.2.3. *Let G be a finite group. Then the following are equivalent:*

(i) *G is nilpotent.*

(ii) *All Sylow subgroups of G are normal.*

(iii) *G is a direct product of its Sylow subgroups.*

(iv) *All subgroups of G are subnormal.*

Definition 1.2.4. Normal series 1.1 are called *chief series* if for all $i \in \{1, \dots, n-1\}$ G_{i+1}/G_i is a minimal normal subgroup of G/G_i. The factor G_{i+1}/G_i is called a *chief factor*. If the order of G_{i+1}/G_i is a power of a prime for some prime p, then G_{i+1}/G_i is called a *p-chief factor*.

Finite groups always have chief series and any two chief series have the same length with their chief factors pairwise isomorphic. One important property finite solvable groups have is that chief factors are *elementary abelian p-groups* for some prime p.

1.3 Lower central series and nilpotent residual

Let G be a group and $x, y \in G$. The *commutator* of x and y is $[x, y] = x^{-1}y^{-1}xy$. If H and K are two subgroups of the group G then define $[H, K] = \langle [h, k] \mid h \in H,\ k \in K \rangle$. The *derived subgroup* of G is defined as $G' = [G, G]$. G/G' is the largest abelian quotient of G, that is if H is a normal subgroup of G and G/H is abelian then G' is a subgroup of H.

Remark 1.3.1. Let H, K be subgroups of the group G.

1. $[H, K] = \langle 1 \rangle$ if and only if $H \leq C_G(K)$ if and only if $K \leq C_G(H)$.
2. $K \leq N_G(H)$ if and only if $[H, K] \leq H$.
3. If H and K are normal subgroups of G then $[H, K]$ is normal subgroup of G and $[H, K] \leq H \cap K$.

Let G be any group and define the following series of subgroups inductively.

$$\begin{aligned}
\gamma_1(G) &= G \\
\gamma_2(G) &= [\gamma_1(G), G] = [G, G] \\
&\vdots \\
\gamma_{i+1}(G) &= [\gamma_i(G), G] = [G, G, \ldots, G].
\end{aligned}$$

The series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_i(G) \geq \cdots \tag{1.2}$$

are called *lower central series of* G. Each $\gamma_i(G)$ is characteristic subgroup of G and $\gamma_i(G)/\gamma_{i+1}(G)$ lies in the center of $G/\gamma_{i+1}(G)$.

Remark 1.3.2. For any finite group G there is an integer n so that

$$\gamma_n(G) = \gamma_{n+1}(G) = \gamma_{n+2}(G) = \cdots .$$

If G is nilpotent then lower central series 1.2 reach $\langle 1 \rangle$. For non-nilpotent groups $\gamma_n(G)$ is a nontrivial subgroup of G.

Suppose there exist an n as in Remark 1.3.2. Let $\gamma_*(G) = \gamma_n(G)$, that is $\gamma_*(G)$ is the smallest term of the lower central series. Then $\gamma_*(G)$ is the smallest normal subgroup of G with $G/\gamma_*(G)$ being nilpotent.

Definition 1.3.3. $\gamma_*(G)$ is called the *nilpotent residual* of G.

The above definition of nilpotent residual is equivalent to the following:

$$\gamma_*(G) = \bigcap\{H \mid H \lhd G \text{ and } G/H \text{ is nilpotent }\}.$$

Since $\gamma_*(G)$ is the smallest normal subgroup of G such that $G/\gamma_*(G)$ is nilpotent and G/G' is the largest abelian quotient, that is G/G' is nilpotent, then $\gamma_*(G) \leq G'$.

1.4 The Fitting subgroup Fit(G) and $O_p(G)$

The subgroup generated by all the normal nilpotent subgroups of a group G is called the *Fitting subgroup* of G, denoted by $\mathrm{Fit}(G)$. For a finite group G the $\mathrm{Fit}(G)$ is the unique largest nilpotent subgroup of G.

Theorem 1.4.1. *If G is a finite group then*

$$\mathrm{Fit}(G) = \bigcap\{C_G(H/K) \mid H/K \textit{ is a chief factor of } G\}.$$

Let G be a group and p a prime. Define

$$O_p(G) = \langle A \mid A \lhd G \text{ and } A \text{ is a } p\text{-group}\rangle.$$

Similarly $O_{p'}(G)$ is defined where A is a p'-group. $O_p(G)$ is the largest normal p-subgroup of G and we know that p-groups are nilpotent. Hence, it is clear that $O_p(G)$ is contained in the $\mathrm{Fit}(G)$. In fact, $O_p(G)$ is the intersection of all the Sylow p-subgroup of G, that is

$$O_p(G) = \bigcap\{P \mid P \in \mathrm{Syl}_p(G)\}.$$

It is easy to see that $O_p(G)$ is the Sylow p-subgroup of the $\mathrm{Fit}(G)$ and that $\mathrm{Fit}(G)$ is the direct product of $O_{p_i}(G)$, where p_i is the prime dividing $|\mathrm{Fit}(G)|$.

1.5 Permutable and Sylow-permutable subgroups.

Definition 1.5.1. Let H be a subgroup of a finite group G.

1. H is *permutable* in G, denoted by H per G, if $HK = KH$ for all subgroups K of G.

2. H is *Sylow-permutable* in G, denoted by H S-per G, if $HP = PH$ for all Sylow subgroups P of G.

It is clear that if H is a normal subgroup of G then H is permutable (Sylow-permutable) in G. Subnormal subgroups in general are not permutable.

Example 1.5.2. Consider a Dihedral group of order 8:

$$D_8 = \langle x, y \mid x^4 = y^2 = 1,\ x^y = x^{-1} \rangle$$

D_8 is nilpotent, thus every subgroup is subnormal. Let $H = \langle y \rangle$ and $K = \langle xy \rangle$. It can be easily verified that $HK \neq KH$.

Other important facts about permutable and Sylow-permutable subgroups are stated below.

Theorem 1.5.3. *Let G be a finite group.*

(1) (Ore, [11]) *If H is maximal permutable subgroup of G then $H \lhd G$.*

(2) (Ore, [11]) *If H per G then H is subnormal subgroup of G.*

(3) (Kegel, [10]) *If $H \leq K \leq G$ and H S-per G then H S-per K.*

(4) (Kegel, [10]) *If H S-per G then H is subnormal subgroup of G.*

1.6 $\mathcal{PST}$, $\mathcal{PT}$ and $\mathcal{T}$ groups

In general, normality, permutability and Sylow-permutability are not transitive relations.

Definition 1.6.1. Groups in which normality, permutability and Sylow-permutability are transitive relations are called $\mathcal{T}$, $\mathcal{PT}$ and $\mathcal{PST}$ groups, respectively.

It is clear that normality is transitive is the same as saying that a subnormal subgroup is normal. In other words, $\mathcal{T}$-groups are precisely the groups in which every subnormal subgroup is normal. Also, Theorem 1.5.3 parts (2) and (4) imply that $\mathcal{PT}$ ($\mathcal{PST}$) groups are precisely the groups in which every subnormal subgroup is permutable (Sylow-permutable). Since normal subgroups are permutable and obviously permutable subgroups are Sylow-permutable then it follows that

$$\mathcal{T} \subset \mathcal{PT} \subset \mathcal{PST}.$$

Also, note that the containment is proper, for a dihedral group of order 8 is a $\mathcal{PST}$-group but not a $\mathcal{PT}$-group and the modular group of order 16 is a $\mathcal{PT}$-group but not a $\mathcal{T}$-group. In addition, nilpotent groups are solvable $\mathcal{PST}$-groups.
Solvable $\mathcal{PST}$, $\mathcal{PT}$ and $\mathcal{T}$ groups were studied and characterized by Agrawal [1], Zacher [17] , and Gaschütz [8]. The following theorem summarizes some of their results.

Theorem 1.6.2. *Let G be a finite group and let L be the nilpotent residual of G.*

(i) **(Agrawal [1])** *G is a solvable $\mathcal{PST}$ group if and only if L is abelian Hall subgroup of G of odd order, and G acts by conjugation as a power automorphism on L.*

(ii) **(Zacher [17])** *G is a solvable $\mathcal{PT}$-group if and only if G is a solvable $\mathcal{PST}$-group and G/L is an Iwasawa group.*

(iii) **(Gaschütz [8])** *G is a solvable $\mathcal{T}$-group if and only if G is a solvable $\mathcal{PST}$-group and G/L is a Dedekind group.*

Definition 1.6.3.

1. A *Dedekind group* is a group in which every subgroup is normal.

2. An *Iwasawa group* is a group in which every subgroup is permutable.

Also, note that if G is a solvable $\mathcal{T}$, $\mathcal{PT}$ or $\mathcal{PST}$ group then every subgroup and every quotient inherits the same properties. The above classes of groups were studied in detail by many authors. Another characterization of solvable $\mathcal{PST}$-groups that will be used later was provided by Alejandre, Ballester-Bolinches, and Pedraza-Aguilera in [2].

Theorem 1.6.4. *Let G be a finite group. Then G is a $\mathcal{PST}$ solvable group if and only if G is supersolvable and all its p-chief factors are isomorphic when regarded as G-modules for every prime p.*

1.7 $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups

Robinson in [12] introduced classes of groups in which cyclic subnormal subgroups are S-permutable, permutable, or normal.

Definition 1.7.1. *A group G is called a $\mathcal{PST}_c$, $\mathcal{PT}_c$ or a $\mathcal{T}_c$-group if every cyclic subnormal subgroup of G is Sylow-permutable, permutable or normal in G, respectively.*

Recall, that $\mathcal{PST}$, $\mathcal{PT}$ and $\mathcal{T}$ groups are defined in terms of transitivity properties. For $\mathcal{PT}_c$ ($\mathcal{T}_c$) groups it is not enough to have a cyclic subgroup to be permutable (normal) in some permutable (normal) subgroup of G. We must have conditions on the Fitting subgroup. An example that demonstrates that conditions on $\mathrm{Fit}(G)$ cannot be omitted can be found in [12] page 174.

Lemma 1.7.2. (Robinson [12]) *Let G be a finite group.*

(i) *G is a $\mathcal{PST}_c$-group if and only if a cyclic subgroup of G is Sylow-permutable in G whenever it is Sylow-permutable in some Sylow-permutable subgroup of G.*

(ii) *G is a $\mathcal{PT}_c$-group if and only if Fit(G) is an Iwasawa group and a cyclic subgroup of G is permutable in G whenever it is permutable in some permutable subgroup of G.*

(iii) *G is a $\mathcal{T}_c$-group if and only if Fit(G) is a Dedekind group and a cyclic subgroup of G is normal in G whenever it is subnormal with defect at most 2.*

The classes of $\mathcal{T}_c$, $\mathcal{PT}_c$ and $\mathcal{PST}_c$-groups are proper subclasses of each other as demonstrated in Example 2.7.1 in Chapter 2. Also, it is easy to see that a $\mathcal{T}$-group is necessarily a $\mathcal{T}_c$-group. Similarly for $\mathcal{PT}$ and $\mathcal{PST}$-groups.

$$\begin{array}{ccccc} \mathcal{T} & \rightarrow & \mathcal{PT} & \rightarrow & \mathcal{PST} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{T}_c & \rightarrow & \mathcal{PT}_c & \rightarrow & \mathcal{PST}_c \end{array}$$

Robinson in [12] provided characterizations for both solvable and insolvable cases, here we mention only the solvable case.

Theorem 1.7.3. (Robinson [12]) *Let G be a finite group and $F = \mathrm{Fit}(G)$.*

(1) *A group G is a solvable $\mathcal{PST}_c$ - group if and only if there is a normal subgroup L such that,*

  (i) *L is abelian and G/L is nilpotent.*

  (ii) *p'-elements of G induce power automorphisms in L_p for all primes p.*

  (iii) $\pi(L) \cap \pi(F/L) = \emptyset$

(2) *A group G is a solvable $\mathcal{PT}_c$ ($\mathcal{T}_c$) - group if and only if G is a solvable $\mathcal{PST}_c$-group such that all elements of G induce power automorphisms in L and F/L is Iwasawa (Dedekind) group, where L is the normal subgroup as described in (1).*

Note the important distinction between solvable $\mathcal{PST}$ and $\mathcal{PST}_c$ groups is that the nilpotent residual is a Hall-subgroup of the Fitting subgroup whereas the nilpotent residual of a solvable $\mathcal{PST}$-group is a Hall subgroup of the entire group.

Remark. ([7], 2.4.1) *Let G be a finite solvable $\mathcal{PST}_c$-group such that the nilpotent residual of G is a Hall-subgroup of G. Then G is a solvable $\mathcal{PST}$-group.*

Also, note that the class of solvable $\mathcal{PST}_c$-groups is not subgroup closed nor quotient closed, however a normal subgroup of a solvable $\mathcal{PST}_c$-group is also a $\mathcal{PST}_c$-group. For details the reader may consult [12] Theorems 2.5 and 2.6. The following example is a solvable $\mathcal{T}_c$ group that is not a $\mathcal{T}$-group.

Example 1.7.4.

$$G = \left\langle a, b, c, d, f \;\middle|\; \begin{array}{l} a^7 = b^3 = c^2 = d^3 = f^5 = (ac)^2 = (bc)^2 = a^5 a^d = 1 \\ [a,b] = [c,d] = [a,f] = [b,f] = [c,f] = [d,f] = [b,d] = 1 \end{array} \right\rangle$$

(i) $G \cong ((C_7 \times C_3) \rtimes (C_2 \times C_3)) \times C_5$.

(ii) The Fitting subgroup of G is $\mathrm{Fit}(G) = (C_7 \times C_3) \times C_5$.

(iii) The nilpotent residual of G is $\gamma_*(G) = C_7 \times C_3$.

(iv) $\gamma_*(G)$ is a Hall-subgroup of the $\mathrm{Fit}(G)$ but not a Hall-subgroup of G.

(v) G is a solvable $\mathcal{T}_c$-group that is not a $\mathcal{T}$-group.

(vi) Let $H = \langle b, c, d \rangle$ be a subgroup of G. $H \cong S_3 \times C_3$. The nilpotent residual $\gamma_*(H) = \langle b \rangle$ is not a Hall subgroup of the $\mathrm{Fit}(H) = \langle b, d \rangle$. Thus, H is not a $\mathcal{T}_c$ group. This demonstrates that solvable $\mathcal{T}_c$ groups are not subgroup closed.

Chapter 2 Direct Products of Certain Classes of Finite Groups

In this chapter we analyze direct products of solvable groups from classes that have been introduced in the previous chapter. In particular, given two groups from one class what are the necessary and sufficient conditions for a direct product to stay in the same class.

Agrawal [1] has shown that *if G_1 and G_2 are $\mathcal{PST}$ groups whose orders are relatively prime, then $G_1 \times G_2$ is a $\mathcal{PST}$-group.* However, it is not necessary for the solvable groups to have relatively prime orders to remain in the class. It turns out that the key lies in the nilpotent residual of the group.

Before we proceed to the next section we establish a similar result as in [1] for $\mathcal{PT}$ and $\mathcal{T}$ groups with relatively prime orders.

Corollary 2.0.5. *Let $G_1, \dots, G_n$ be finite groups such that $(|G_i|, |G_j|) = 1$ for all $i, j \in \{1, \dots, n\}$ and $i \neq j$. Then,*

(i) *$G_1, \dots, G_n \in \mathcal{PT}$ if and only if $G_1 \times \cdots \times G_n \in \mathcal{PT}$.*

(ii) *$G_1, \dots, G_n \in \mathcal{T}$ if and only if $G_1 \times \cdots \times G_n \in \mathcal{T}$.*

Proof. (i). Suppose $G_1, \dots, G_n \in \mathcal{PT}$ and let $G := G_1 \times \cdots \times G_n$. Let H be a subnormal subgroup of G and K any subgroup of G. Since $(|G_i|, |G_j|) = 1$ for all $i, j \in \{1, \dots, n\}$ then $H = H_1 \times \cdots \times H_n$ and $K = K_1 \times \cdots \times K_n$, where H_i is subnormal subgroup of G_i and K_i is a subgroup of G_i. Since each G_i is a $\mathcal{PT}$-group then H_i and K_i permute. Also, for $i \neq j$, G_j centralizes G_i hence H_i and K_j permute. Therefore, $HK = KH$ for all subgroups K of G. Thus, G is a $\mathcal{PT}$-group. As for the other direction, since normal subgroups of $\mathcal{PT}$-groups inherit the same properties then each G_i is a $\mathcal{PT}$ group.

(ii). Suppose $G_1, \dots, G_n \in \mathcal{T}$ and let $G := G_1 \times \cdots \times G_n$. Let H be a subnormal subgroup of G. Since $(|G_i|, |G_j|) = 1$ for all $i, j \in \{1, \dots, n\}$ then $H = H_1 \times \cdots \times H_n$, where H_i is subnormal subgroup of G_i. Since each G_i is a $\mathcal{T}$-group then H_i is normal subgroup of G_i. Therefore $H = H_1 \times \cdots \times H_n$ is normal subgroup of G. Thus, G is a $\mathcal{T}$-group. As for the other direction, since normal subgroups of $\mathcal{T}$-groups inherit the same properties then each G_i is a $\mathcal{T}$ group.

□

2.1 Direct products of solvable $\mathcal{PST}$-groups

Two $\mathcal{PST}$, $\mathcal{PT}$ or $\mathcal{T}$ groups with relatively prime orders stay in the same class, however there are numerous examples of solvable $\mathcal{PST}$, $\mathcal{PT}$ or $\mathcal{T}$ groups that suggest otherwise. Consider a few of these examples.

Example 2.1.1. In this example S_3 denotes a symmetric group of order 6, C_2 and C_3 are cyclic groups of order 2 and 3, respectively, and D_{10} a dihedral group of order 10.

(i) Let $G := S_3 \times C_2$ with the following presentation:

$$\langle x, y, z \mid x^3 = y^2 = z^2 = 1,\ (xy)^2 = 1,\ [x, z] = [y, z] = 1\rangle.$$

Nilpotent residual of $S_3 \times C_2$ is $\gamma_*(G) = \langle x\rangle$, which is the nilpotent residual of S_3. Both S_3 and C_2 are solvable $\mathcal{PST}$-groups.
Clearly, $\gamma_*(G)$ is abelian Hall subgroup of G of odd order. Also, it is easy to see that every subgroup of $\gamma_*(G)$ is normal in $S_3 \times C_2$. Thus, $S_3 \times C_2$ is a $\mathcal{PST}$-group. Notice, the orders of direct factors are not relatively prime but the order of $\gamma_*(G)$ is relatively prime to the order of C_2.

(ii) Let $G := S_3 \times D_{10}$. This group has the following presentation,

$$\left\langle x, y, z, w \;\middle|\; \begin{array}{l} x^3 = y^2 = z^5 = w^2 = (xy)^2 = (zw)^2 = 1 \\ [x, z] = [x, w] = [y, z] = [y, w] = 1 \end{array} \right\rangle.$$

The nilpotent residual of G is $\langle x\rangle \times \langle z\rangle$, where $\langle x\rangle$ is the nilpotent residual of S_3 and $\langle z\rangle$ is the nilpotent residual of D_{10}. Both S_3 and D_{10} are solvable $\mathcal{PST}$-groups, and by a direct computation one can determine that G is a solvable $\mathcal{PST}$-group.
Notice again, that the orders of a direct factors are not relatively prime, whereas the order of $\langle x\rangle$ is relatively prime to the order of D_{10} and the order of $\langle z\rangle$ is relatively prime to the order of S_3.

(iii) Let $H := S_3 \times C_3$. H is not a $\mathcal{PST}$-group whereas each direct factor is a $\mathcal{PST}$-group. The nilpotent residual of H is just an alternating group of order 3 and its order is not relatively prime to the order of C_3.

A closer analysis of the characterization of solvable $\mathcal{PST}$-groups and the examples motivate the following result.

Theorem 2.1.2. *Let G_1 and G_2 be finite groups. $G_1 \times G_2$ is a solvable $\mathcal{PST}$-group if and only if G_1 and G_2 are solvable $\mathcal{PST}$ groups and $(|\gamma_*(G_i)|, |G_j|) = 1$ for $i \neq j$ and $i, j \in \{1, 2\}$.*

Proof. First note that $\gamma_*(G_1 \times G_2) = \gamma_*(G_1) \times \gamma_*(G_2)$. Let $L := \gamma_*(G_1 \times G_2) = L_1 \times L_2$, where $L_i = \gamma_*(G_i)$ for $i \in \{1, 2\}$.
Suppose $G_1 \times G_2$ is a solvable $\mathcal{PST}$ group. It is clear that G_1 and G_2 are solvable $\mathcal{PST}$ groups. Let $L_p \in \mathrm{Syl}_p(L)$. Then $L_p = L_p^1 \times L_p^2$ where $L_p^i \in \mathrm{Syl}_p(L_i)$ for $i \in \{1, 2\}$. Suppose L_p^1 and L_p^2 are both not trivial and take $x = x_1 x_2 \in L_p$ so that $x_i \neq 1$, where $x_i \in L_p^i$ and $|x_i| = p^{\alpha_i}$ for $i \in \{1, 2\}$. Let $g \in G_1$ with $g \neq 1$. Then g acts by conjugation as a power automorphism on x_1 and trivially on x_2, that is $(x_1 x_2)^g = x_1^m x_2$ for some positive integer m. But g also acts as a power automorphism on x, i.e. $(x_1 x_2)^g = (x_1 x_2)^n$ for some positive integer n. Thus, we get that $x_1^m x_2 = x_1^n x_2^n$. This means that $m \equiv n \bmod p^{\alpha_1}$ and $1 \equiv n \bmod p^{\alpha_2}$.
If $\alpha_1 \geq \alpha_2$ then $m \equiv 1 \bmod p^{\alpha_2}$, that is $x_2^g = x_2^m$. But G_1 centralizes G_2 and $x_2 \neq 1$, thus $m = 1$. If $\alpha_1 \leq \alpha_2$ then $1 \equiv m \bmod p^{\alpha_1}$. This means that $x_1^g = x_1$. Since $x_1 \neq 1$ then $m = 1$. In both cases we get that $m = 1$, which implies that G_1 acts trivially on L_p^1. Now, $L_p^1 = [L_p^1, G_1] = \langle 1 \rangle$, a contradiction. Therefore, either L_p^1 is trivial or L_p^2 is trivial, that is $(|L_1|, |L_2|) = 1$.
Since L is a Hall subgroup of $G_1 \times G_2$, then

$$(|L|, (G_1 \times G_2 : L)) = (|L_1||L_2|, (G_1 : L_1)(G_2 : L_2)) = 1.$$

Hence, the above implies that

$$(|L_i|, |L_j|(G_i : L_i)(G_j : L_j)) = 1,$$

which means $(|L_i|, |G_j|) = 1$ for $i \neq j$ and $i, j \in \{1, 2\}$.
Conversely, suppose that G_1 and G_2 are solvable $\mathcal{PST}$-groups and $(|L_i|, |G_j|) = 1$ for $i \neq j$ and $i, j \in \{1, 2\}$. It is clear that $G_1 \times G_2$ is solvable. In addition, since each G_i, $i \in \{1, 2\}$, is supersolvable it follows that $G_1 \times G_2$ is supersolvable. Next, $(G_1 \times G_2)/L$ is nilpotent, thus every p-chief factor of $G_1 \times G_2$ above L is central, where p is some prime dividing the order of $G_1 \times G_2$. Hence, if H_1/K_1 and H_2/K_2 are two p-chief factors of $G_1 \times G_2$ above L, then for all $g \in G_1 \times G_2$ and $h_i K_i \in H_i/K_i$, $h_i^g K_i = h_i K_i$, for $i \in \{1, 2\}$, that is g acts in the same way on $h_1 K_1$ and $h_2 K_2$. Therefore, all p-chief factors above L are $(G_1 \times G_2)$-isomorphic. Note, that $(|L_i|, |G_j|) = 1$ for $i \neq j$ implies $(|L_1|, |L_2|) = 1$, then every p-chief factor of $G_1 \times G_2$ below $L = L_1 \times L_2$ is a p-chief factor of either G_1 or G_2. Since G_i is a $\mathcal{PST}$-group then all p-chief factors of

G_i are G_i-isomorphic, for $i \in \{1,2\}$. In particular, p-chief factors of $G_1 \times G_2$ below $L = L_1 \times L_2$ are $(G_1 \times G_2)$-isomorphic. Therefore, $G_1 \times G_2$ is a $\mathcal{PST}$-group by Theorem 1.2. This concludes the proof. $\square$

Theorem 2.1.2 is true for a direct product of finitely many solvable $\mathcal{PST}$-groups. Let $G := G_1 \times G_2 \times \cdots \times G_n$ and suppose that $(|\gamma_*(G_i)|, |G_j|) = 1$ for $1 \leq i \neq j \leq n$. Then

$$(|\gamma_*(G_i)|, |G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n|) = 1 \tag{2.1}$$

and

$$(|\gamma_*(G_1 \times \cdots \times G_{j-1} \times G_{j+1} \times \cdots \times G_n)|, |G_j|) = 1. \tag{2.2}$$

Corollary 2.1.3. *Let $G_1, \ldots, G_n$ be finite groups. $G_1 \times G_2 \times \cdots \times G_n$ is a solvable $\mathcal{PST}$ group if and only if*
$G_1, G_2, \ldots, G_n$ are solvable $\mathcal{PST}$ groups, and $(|\gamma_*(G_i)|, |G_j|) = 1$ for $1 \leq i, j \leq n$, $i \neq j$ and $n \geq 2$.

The proof of Corollary 2.1.3 is by induction on n with a direct application of Theorem 2.1.2.

Proof. The solvability is clear. Next, suppose that the statement is true for $n-1$ and let $G := G_1 \times \cdots \times G_{n-1}$. Then, Theorem 2.1.2 and 2.1,2.2 imply that $G \times G_n$ is a solvable $\mathcal{PST}$-group if and only if G and G_n are solvable $\mathcal{PST}$-groups such that $(|\gamma_*(G)|, |G_n|) = 1$ and $(|G|, |\gamma_*(G_n)|) = 1$. $\square$

Now the question arises whether Theorem 2.1.2 could be applied to the $\mathcal{T}$ and $\mathcal{PT}$ groups and what are the additional conditions needed in order to stay in the class.

2.2 Direct products of Iwasawa and Dedekind groups

The extension of Theorem 2.1.2 to solvable $\mathcal{PT}$ and $\mathcal{T}$ groups will depend on our understanding of direct products of Iwasawa and Dedekind p-groups.

Definition 2.2.1.

(1) A *Dedekind group* is a group in which every subgroup is normal.

(2) An *Iwasawa group* is a group in which every subgroup is permutable.

The reader should note that Iwasawa (Dedekind) groups are necessarily solvable $\mathcal{PT}$ ($\mathcal{T}$) groups, respectively.

Theorem 2.2.2. (Agrawal [1]) *Let G be a finite $\mathcal{PST}$-group.*

(1) *If all Sylow subgroups of G are Iwasawa, then G is a $\mathcal{PT}$-group.*

(2) *If all Sylow subgroups of G are Dedekind, then G is a $\mathcal{T}$-group.*

Let G_1 and G_2 be solvable $\mathcal{PT}$-groups and $(|\gamma_*(G_i)|, |G_j|) = 1$ for $i \neq j$. Since the class of $\mathcal{PT}$-groups is contained in the class of $\mathcal{PST}$-groups then from Theorem 2.1.2 we immediately get that $G_1 \times G_2$ is a solvable $\mathcal{PST}$-group. Also, solvable $\mathcal{PT}$-groups are subgroup-closed, hence every Sylow subgroup of G_i is a $\mathcal{PT}$-group, that is an Iwasawa-group. In particular, if P is a Sylow subgroup of $G_1 \times G_2$ then it is a direct product of two Iwasawa groups. Similar ideas apply to $\mathcal{T}$ groups. Therefore, we consider at first when a direct product of Iwasawa (Dedekind) groups is again an Iwasawa (Dedekind) group.
The following theorems will be used in order to establish results of this section.

Theorem 2.2.3 (Iwasawa, 1941 [15], page 55). *A finite p-group G has modular subgroup lattice if and only if*

(a) *G is a direct product of a Q_8 with an elementary abelian 2-group, or*

(b) *G contains an abelian normal subgroup A with cyclic factor group G/A; further there exists an element $b \in G$ with $G = A\langle b\rangle$ and s positive integer such that $a^b = a^{1+p^s}$ for all $a \in A$, with $s \geq 2$ in case $p = 2$.*

Note, that a finite p-group P has *modular subgroup lattice* if and only if all subgroups of P are permutable that is if and only if P is an Iwasawa-group. The details can be found in [15].

Remark 2.2.4. The rest of this thesis will assume the notation of Theorem 2.2.3 whenever there is a finite non abelian Iwasawa p-group. In particular, if P_i is non abelian Iwasawa p-group and Q_8 is not contained in P_i, in case $p = 2$, then $P_i = A_i\langle x_i\rangle$ where A_i is abelian normal subgroup of P_i and P_i/A_i is cyclic; furthermore there is a positive integer s_i such that $a^{x_i} = a^{1+p^{s_i}}$ for all $a \in A_i$ with $s_i \geq 2$ in case $p = 2$.

Theorem 2.2.5 (Dedekind-Baer, [13], page 139). *All the subgroups of a group G are normal if and only if*

(a) *G is abelian, or*

(b) *G is the direct product of a Q_8, an elementary abelian 2-group and an abelian group with all its elements of odd order.*

The following remark is a restriction of Corollary 2.0.5 to p-groups.

Remark 2.2.6. Let $G_1, G_2, \ldots, G_n$ be finite p_i -groups, where p_i is a prime and $p_i \neq p_j$ for all $i, j \in \{1, \ldots, n\}$. Then,

(i) $G_1, \ldots, G_n \in \mathcal{PT}$ if and only if $G_1 \times \cdots \times G_n \in \mathcal{PT}$.

(ii) $G_1, \ldots, G_n \in \mathcal{T}$ if and only if $G_1 \times \cdots \times G_n \in \mathcal{T}$.

In particular, Remark 2.2.6 implies that if P is a p-group and Q is a q-group for some primes $p \neq q$ such that P and Q are Iwasawa (Dedekind) groups, then direct product, $P \times Q$, is Iwasawa (Dedekind) group respectively. Direct products of Dedekind p-groups, for the same prime p, will follow directly from Dedekind-Baer Theorem 2.2.5 and present no difficulty, that is we only have to be careful about Sylow 2-group.
A more interesting case arises, that requires some investigation, is when a p-group P is a direct product of two or more Iwasawa p-groups and Q_8 is not contained in P.

Example 2.2.7. Consider a Modular group of order 16

$$G_1 = \langle x, y | x^8 = y^2 = 1, x^y = x^5 \rangle$$

Note, that G_1 is an Iwasawa group.

(i) Let $G := G_1 \times C_2$. Then G is an Iwasawa group.

(ii) Let $G := G_1 \times C_8$ where $C_8 = \langle z \rangle$. Then $\langle xz \rangle$ is not permutable in G. Thus, G is not an Iwasawa group. Also, note that $\mathrm{Exp}(C_8)$ is greater than 2^2.

(iii) Let $G := G_1 \times G_1$. Then G is not an Iwasawa since $\langle x^i x^j \rangle$ does not permute with $\langle y \rangle$.

Example 2.2.7 shows that a direct product of two Iwasawa groups is not necessarily an Iwasawa even if one factor is abelian and the other is not.
The next Theorem assumes the notation of Remark 2.2.4.

Theorem 2.2.8. *Let $P = P_1 \times P_2$ be a non abelian finite p-group and $Q_8 \nleq P$, in case $p = 2$. Then, P is an Iwasawa group if and only if P_1 and P_2 are Iwasawa and either for $i = 1$ or $i = 2$ P_i is abelian such that $\mathrm{Exp}(P_i) \leq p^s$ and s is the integer that comes from the non abelian factor P_j for $j \neq i$.*

Proof. Suppose P is an Iwasawa group but P_1 and P_2 are both not abelian. It is clear that each P_i is an Iwasawa group, thus $P_i = A_i\langle x_i\rangle$ where A_i is abelian normal subgroup of P_i such that $a^{x_i} = a^{1+p^{s_i}}$ for all $a \in A_i$ with $s_i \geq 2$. Also, notice that p^{s_i} must be strictly less than $\mathrm{Exp}(A_i)$, otherwise $a_i^{x_i} = a_i^{1+p^{s_i}} = a_i a_i^{p^{s_i}} = a_i a_i^{\mathrm{Exp}(A_i)\ell} = a_i$ for some positive integer ℓ, i.e. x_i will act trivially on all $a_i \in A_i$.
Let $a_i \in A_i$ such that $a_i \neq 1$. Since P is an Iwasawa then all subgroups of P are permutable. This means that $\langle a_1a_2\rangle$ permutes with $\langle x_1\rangle$, that is $a_1a_2x_1$ must be an element of $\langle x_1\rangle\langle a_1a_2\rangle$. Hence,

$$a_1a_2x_1 = x_1^k(a_1a_2)^\ell = x_1^k a_1^\ell a_2^\ell$$

for some positive integers k and ℓ. But,

$$a_1a_2x_1 = x_1x_1^{-1}a_1x_1x_1^{-1}a_2x_1 = x_1a_1^{1+p^{s_1}}a_2$$

$$x_1^k a_1^\ell a_2^\ell = x_1a_1^{1+p^{s_1}}a_2,$$

which implies that either $\ell = 1$ or $\ell = 1 + p^{s_1}$. Since x_1 does not act trivially on A_1, then it must be that $\ell = 1 + p^{s_1}$. Therefore, $a_2 = a_2^{1+p^{s_1}}$ for all $a_2 \in A_2$, and as a result the $\mathrm{Exp}(A_2) \leq p^{s_1}$. Similarly, $\langle a_1a_2\rangle$ must permute with $\langle x_2\rangle$, and hence by a similar argument $\mathrm{Exp}(A_1) \leq p^{s_2}$.
Now, for $\langle a_1a_2\rangle$ to permute with $\langle x_1\rangle$ and $\langle x_2\rangle$ it must be that $\mathrm{Exp}(A_2) \leq p^{s_1}$ and $\mathrm{Exp}(A_1) \leq p^{s_2}$. Recall, that p^{s_i} must be strictly less that the exponent of A_i. Therefore,

$$p^{s_1} < \mathrm{Exp}(A_1) \leq p^{s_2} < \mathrm{Exp}(A_2) \leq p^{s_1},$$

in particular $p^{s_i} < p^{s_i}$ which is a contradiction. Hence $\mathrm{Exp}(A_2) \leq p^{s_1}$ and $\mathrm{Exp}(A_1) \leq p^{s_2}$ cannot happen at the same time, therefore for either $i = 1$ or $i = 2$ the x_i must act trivially on A_i, i.e. $P_i = A_i \times \langle x_i\rangle$ is abelian.
The $\mathrm{Exp}(P_i) \leq p^{s_j}$ for $i \neq j$ follows by a similar argument. In particular, without loss of generality suppose that P_2 is abelian. Let $a_2 \in P_2$ and $a_1 \in A_1$, where A_1 as above. Then $\langle a_1a_2\rangle$ and $\langle x_1\rangle$ must permute, that is $a_1a_2x_1$ must be an element of $\langle x_1\rangle\langle a_1a_2\rangle$. Hence,

$$x_1^k a_1^\ell a_2^\ell = x_1a_1^{1+p^{s_1}}a_2,$$

which implies that either $\ell = 1$ or $\ell = 1 + p^{s_1}$. Since x_1 does not act trivially on A_1, then it must be that $\ell = 1 + p^{s_1}$. Therefore, $a_2 = a_2^{1+p^{s_1}}$ for all $a_2 \in P_2$, and as a result the $\mathrm{Exp}(P_2) \leq p^{s_1}$.
Conversely, suppose that P_1 is abelian and let $A = P_1 \times A_2$. Then A is abelian normal

subgroup of $P_1 \times P_2$ and $(P_1 \times P_2)/A$ is cyclic. Let $a \in A$ then $a = gh$ where $g \in P_1$ and $h \in A_2$. Then

$$a^x = (gh)^x = gh^x = gh^{1+p^{s_2}}.$$

Since $\mathrm{Exp}(P_1) \leq p^{s_2}$ then $\mathrm{ord}(g) \leq p^{s_2}$ that is $g^{p^{s_2}} = 1$ which implies that $g^{1+p^{s_2}} = g$. Therefore,

$$a^x = gh^{1+p^{s_2}} = g^{1+p^{s_2}} h^{1+p^{s_2}} = a^{1+p^{s_2}}.$$

Hence $P_1 \times P_2$ is an Iwasawa group. This concludes the proof. $\square$

Theorem 2.2.8 implies that if $P := P_1 \times \cdots \times P_n$ is a non abelian Iwasawa p-group and Q_8 is not contained in P, then exactly one P_i will be non abelian Iwasawa p-group and $P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n$ must be abelian p-group such that $\mathrm{Exp}(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n) \leq p^{s_i}$.
Also, note that if G is any Iwasawa group then every subgroup is permutable in G and consequently subnormal which implies that G is nilpotent and thus a direct product of its Sylow subgroups. Hence, a general Iwasawa group G is a direct product of unique maximal Iwasawa p-groups.

2.3 Direct products of solvable $\mathcal{PT}$ and $\mathcal{T}$ groups

The results up to this point provide conditions needed in order to extend Theorem 2.1.2 to solvable $\mathcal{PT}$ and $\mathcal{T}$ groups.

Remark 2.3.1. The next theorems will employ the following notation.
Let $P := P_1 \times \cdots \times P_n$ for $n \geq 2$ then $P \backslash P_i := P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n$.

Theorem 2.3.2. *Let $G := G_1 \times \cdots \times G_n$ be a finite group and $P := P_1 \times \cdots \times P_n \in \mathrm{Syl}_p(G)$ where $P_i \in \mathrm{Syl}_p(G_i)$ for some prime p dividing the order of G and $n \geq 2$. Then G is a solvable $\mathcal{PT}$-group if and only if the following hold:*

(i) *$G_1, \ldots, G_n$ are solvable $\mathcal{PT}$-groups such that $(|\gamma_*(G_i)|, |G_j|) = 1$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.*

(ii) *If Q_8 is contained in P, then it is a subgroup of exactly one P_i and $P \backslash P_i$ is elementary 2-abelian or trivial.*

(iii) *If Q_8 is not contained in P, in case $p = 2$, and P is not abelian, then exactly one P_i is non abelian and $P \backslash P_i$ is an abelian p-group such that $\mathrm{Exp}(P \backslash P_i) \leq p^{s_i}$.*

Proof. First suppose that G is a solvable $\mathcal{PT}$-group. Since each G_i is normal in G then $G_1, \ldots, G_n$ are solvable $\mathcal{PT}$-groups. Also, note that G is a solvable $\mathcal{PST}$-group, thus by Corollary 2.1.3 $(|\gamma_*(G_i)|, |G_j|) = 1$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. This proves (i).
Since solvable $\mathcal{PT}$-groups are subgroup closed then P is a solvable $\mathcal{PT}$-group, i.e. P is an Iwasawa group and consequently each P_i is an Iwasawa-group. If Q_8 is contained in P then by Theorem 2.2.3 part (a) P is a direct product of Q_8 and an abelian 2-group, which shows (ii). If Q_8 is not contained in P and P is not abelian then Theorem 2.2.8 implies (iii).
Conversely, suppose (i)-(iii). The solvability of G is clear. Note that G_i are solvable $\mathcal{PST}$-groups for all i, hence (i) implies that G is a solvable $\mathcal{PST}$-group. Also, since G_i are solvable $\mathcal{PT}$-groups then each P_i is an Iwasawa group, in particular P is a direct product of Iwasawa groups. If P is abelian then P is an Iwasawa. If Q_8 is contained in P then (ii) satisfies the hypothesis of Theorem 2.2.3 (a), i.e. P is an Iwasawa. If Q_8 is not contained in P and P is not abelian then (iii) and Theorem 2.2.8 imply that P is an Iwasawa. This means that all Sylow subgroups of G are Iwasawa and since G is a $\mathcal{PST}$-group then by [1] G is a $\mathcal{PT}$-group. This concludes the proof. □

Theorem 2.3.3. *Let $G := G_1 \times \cdots \times G_n$ be a finite group and $P := P_1 \times \cdots \times P_n \in \mathrm{Syl}_p(G)$ where $P_i \in \mathrm{Syl}_p(G_i)$ for some prime p dividing the order of G and $n \geq 2$. Then G is a solvable $\mathcal{T}$-group if and only if the following hold:*

(i) *$G_1, \ldots, G_n$ are solvable $\mathcal{T}$-groups such that $(|\gamma_*(G_i)|, |G_j|) = 1$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.*

(ii) *if Q_8 is contained in P, then it is a subgroup of exactly one P_i and $P \backslash P_i$ is elementary 2-abelian or trivial; otherwise P is abelian.*

Proof. First suppose that G is a solvable $\mathcal{T}$-group. It is immediate from the proof of Theorem 2.3.2 that (i) is satisfied and P is Dedekind group and consequently each P_i is Dedekind-group. If Q_8 is contained in P then by Theorem 2.2.5 part (b) P is a direct product of Q_8 and an abelian 2-group. If Q_8 is not contained in P then P must be abelian. Thus (ii) is proved.
Conversely, suppose (i) and (ii). The solvability of G is clear and since the G_i are solvable $\mathcal{PST}$-groups for all i then (i) implies that G is a solvable $\mathcal{PST}$-group. Also, since each G_i is a solvable $\mathcal{T}$-group then each P_i is Dedekind group, in particular P is a direct product of Dedekind groups. If p is odd prime, then each P_i is abelian,

hence P is abelian. If $p = 2$ then (ii) implies that either P is abelian or a direct product of Q_8 and an elementary abelian 2-group. Hence, all Sylow subgroups of G are Dedekind and since G is a $\mathcal{PST}$-group then by [1] G is a $\mathcal{T}$-group. This concludes the proof. □

2.4 Direct products of $\mathcal{PST}_c$ groups

We begin this section by revisiting the important distinction between solvable $\mathcal{PST}$ and $\mathcal{PST}_c$. Recall, the nilpotent residual of a solvable $\mathcal{PST}_c$-group is a Hall-subgroup of the Fitting subgroup whereas the nilpotent residual of solvable $\mathcal{PST}$-groups is a Hall subgroup of the entire group. Also, bear in mind that the class of solvable $\mathcal{PST}_c$-groups is not subgroup closed nor quotient closed class, however a normal subgroup of a solvable $\mathcal{PST}_c$-group is also a $\mathcal{PST}_c$-group.

Remark 2.4.1. *Let G be a finite solvable $\mathcal{PST}_c$-group such that the nilpotent residual of G is a Hall-subgroup of G. Then G is a solvable $\mathcal{PST}$-group.*

Proof. Let L be a nilpotent residual of G. By Theorem 1.1 in [12] L is abelian of odd order. Thus, all it remains to show that G acts by conjugation as a power automorphism on L.
Let $x \in L$ such that x is not trivial. Since L is abelian then $\langle x \rangle$ is normal subgroup of L, i.e. $\langle x \rangle$ is subnormal subgroup of G. But G is a $\mathcal{PST}_c$, hence $\langle x \rangle$ is S-permutable in G. Let $\pi := \pi(L)$ be the set of primes that divide the order of L and let $q \in \pi'$. Consider a Sylow q subgroup of G, G_q. Then $\langle x \rangle G_q = G_q \langle x \rangle$. Also, $\langle x \rangle$ is a subnormal Hall π-subgroup of $G_q\langle x \rangle$, thus $\langle x \rangle$ is normal subgroup of $G_q\langle x \rangle$, that is G_q normalizes $\langle x \rangle$ and hence $O^\pi(G)$ normalizes $\langle x \rangle$, where $O^\pi(G)$ is the subgroup of G generated by all Sylow q subgroups of G, $q \in \pi'$. Since, L is a Hall π-subgroup then $G = LO^\pi(G)$. But L is nilpotent and so $G/O^\pi(G)$ is nilpotent. Also, L is the smallest normal subgroup of G such that G/L is nilpotent, then $L \leq O^\pi(G)$. Thus, $G = O^\pi(G)$ normalizes $\langle x \rangle$, i.e. G acts by conjugation on L as a power automorphism. Therefore, G is a solvable $\mathcal{PST}$-group. □

We now extend Theorem 2.1.2 to a direct product of $\mathcal{PST}_c$-groups.

Theorem 2.4.2. *Let G_1 and G_2 be finite groups. $G_1 \times G_2$ is a solvable $\mathcal{PST}_c$ group if and only if G_1 and G_2 are solvable $\mathcal{PST}_c$ groups such that $(|\gamma_*(G_i)|, |\mathrm{Fit}(G_j)|) = 1$ for $i \neq j$ and $i, j \in \{1, 2\}$.*

Proof. Let $L := \gamma_*(G_1 \times G_2)$ and $F := \mathrm{Fit}(G_1 \times G_2)$. Then $L = L_1 \times L_2$ and $F = F_1 \times F_2$ where L_i is nilpotent residual of G_i, and F_i is Fitting subgroup of G_i.
First suppose that $G_1 \times G_2$ is a solvable $\mathcal{PST}_c$ group. Since G_i is normal in $G_1 \times G_2$ then each G_i is a solvable $\mathcal{PST}_c$ group. Let $L_p \in Syl_p(L)$, then $L_p = L_p^1 \times L_p^2$. Following the same argument as in Theorem 2.1.2 either L_p^1 or L_p^2 is trivial. Hence $(|L_1|, |L_2|) = 1$, which implies that $(|L_i|, |F_j|) = 1$ for $i \neq j$.
As for the other direction, suppose that G_1 and G_2 are solvable $\mathcal{PST}_c$ groups such that $(|L_i|, |F_j|) = 1$ for $i \neq j$. Trivially, $G_1 \times G_2$ is solvable, L is abelian and $G_1 \times G_2/L$ is nilpotent.
Since L_i is a Hall subgroup of F_i and F_i is a solvable $\mathcal{PST}$-group, then using similar ideas as in the proof of Theorem 2.1.2 we get that L is a Hall subgroup of F and $(|L_1|, |L_2|) = 1$. Thus, all we need to show that p'-elements of $G_1 \times G_2$ induce power automorphisms in $L_p \in Syl_p(L)$ for all primes p.
Let $L_p \in Syl_p(L)$ and $Q \in Syl_q(G_1 \times G_2)$ for $p \neq q$. Then $Q = Q_1 \times Q_2$ where $Q_i \in Syl_q(G_i)$. Since $(|L_1|, |L_2|) = 1$ then $L_p \leq L_1$ or $L_p \leq L_2$. Without loss of generality suppose $L_p \leq L_1$.
Let $a \in L_p$ not trivial, then $\langle a \rangle$ is subnormal subgroup of G_1 and G_1 is $\mathcal{PST}_c$. This implies that $\langle a \rangle$ S-per G_1, in particular $\langle a \rangle Q_1 = Q_1 \langle a \rangle$. Also, G_1 centralizes G_2, then $\langle a \rangle$ permutes with Q_2. Thus, $\langle a \rangle$ permutes with $Q = Q_1 \times Q_2$.
Hence

$$a^Q = a^Q \cap (\langle a \rangle Q) = \langle a \rangle (a^Q \cap Q)$$

But a^Q is a p-group implies that $a^Q \cap Q$ is trivial. Therefore,

$$a^Q = \langle a \rangle$$

Thus we conclude that $G_1 \times G_2$ is a solvable $\mathcal{PST}_c$ - group and the proof is complete. $\square$

Using ideas similar to those in Corollary 2.1.3, one can show by induction that Theorem 2.4.2 is true for a direct product of finitely many solvable $\mathcal{PST}_c$-groups.

2.5 Direct products of $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups

Recall, Agrawal[1], 2.2.2 has shown that a $\mathcal{PST}$ group G is a $\mathcal{PT}$ ($\mathcal{T}$) group if all of its Sylow subgroups are *Iwasawa* (*Dedekind*) respectively. A similar result can be proved for $\mathcal{PST}_c$-groups.

Theorem 2.5.1. *Let G be a finite $\mathcal{PST}_c$-group.*

(1) *If all Sylow subgroups of G are Iwasawa, then G is a $\mathcal{PT}_c$-group.*

(2) *If all Sylow subgroups of G are Dedekind, then G is a $\mathcal{T}_c$-group.*

Proof. (1). Let H be any cyclic subnormal subgroup of the group G and H_p be a direct factor of H for some prime p dividing the order of H. Then H_p is a subgroup of the $F_p \in \mathrm{Syl}_p(\mathrm{Fit}(G))$. Since $F_p = \cap\{G_p \mid G_p \in \mathrm{Syl}_p(G)\}$ then H_p is contained in every Sylow p-subgroup of G. Let $g \in G$ be non trivial, then $\langle g\rangle = \langle g_1\rangle \times \langle g_2\rangle$ where g_1 is a p-element and g_2 is a p'-element. Since H_p is subnormal subgroup of G and G is a $\mathcal{PST}_c$-group then by Lemma 2.2 in [12] p'-elements induce power automorphisms in H_p. In particular, g_2 and H_p permute. Also, by hypothesis each G_p is Iwasawa and $\langle g_1\rangle$ is contained in some Sylow p-subgroup of G, then H_p and $\langle g_1\rangle$ permute. Hence, H_p and $\langle g\rangle$ permute. Since, H is a direct product of cyclic subgroups of prime power order then H and $\langle g\rangle$ permute, which implies that H is permutable in G, that is G is a $\mathcal{PT}_c$-group.
(2). Let H be any cyclic subnormal subgroup of the group G and H_p be a direct factor of H for some prime p dividing the order of H. Then H_p is a subgroup of the $F_p \in \mathrm{Syl}_p(\mathrm{Fit}(G))$. In particular, H_p is contained in every Sylow p-subgroup of G. Since, every Sylow subgroup of G is Dedekind then H_p is normal subgroup of every Sylow p-subgroup. Also, p'-elements induce power automorphisms in H_p, that is H_p is normalized by every p'-element. Thus, we conclude that H_p is normal subgroup of G, which implies that H is normal in G, that is G is a $\mathcal{T}_c$-group. $\square$

Theorem 2.5.1 suggests that Theorems 2.3.2 and 2.3.3 might be extended to $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups, however these classes of groups are not subgroup closed. In particular, it is not clear if a Sylow subgroup of the solvable $\mathcal{PT}_c$ ($\mathcal{T}_c$)-group will inherit the same properties as the group, that is Sylow subgroups are not necessarily Iwasawa (Dedekind).

Open Question. Given a finite solvable $\mathcal{PT}_c$ ($\mathcal{T}_c$)-group, identify the structure of the Sylow subgroups.

There are countless examples of solvable $\mathcal{PT}_c$ and $\mathcal{T}_c$-groups in which Sylow subgroups are all Iwasawa and Dedekind, respectively. Thus, it makes sense to make a restriction to such groups, that is we restrict to solvable $\mathcal{PT}_c$ ($\mathcal{T}_c$) groups in which every Sylow subgroup is Iwasawa (Dedekind).

Corollary 2.5.2. *Let $G := G_1 \times \cdots \times G_n$ be a finite group and $P := P_1 \times \cdots \times P_n \in \mathrm{Syl}_p(G)$ where $P_i \in \mathrm{Syl}_p(G_i)$ for some prime p dividing the order of G and $n \geq 2$. Suppose every Sylow subgroup of G_i is an Iwasawa group, for all $i \in \{1, \ldots, n\}$. Then G is a solvable $\mathcal{PT}_c$-group if the following hold:*

(i) *$G_1, \ldots, G_n$ are solvable $\mathcal{PT}_c$-groups such that $(|\gamma_*(G_i)|, |\mathrm{Fit}(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$.*

(ii) *If Q_8 is contained in P, then it is a subgroup of exactly one P_i and $P \backslash P_i$ is elementary 2-abelian or trivial.*

(iii) *If Q_8 is not contained in P and P is not abelian, then exactly one P_i is non abelian and $P \backslash P_i$ is abelian p-group such that $\mathrm{Exp}(P \backslash P_i) \leq p^{s_i}$.*

Proof. Suppose (i)-(iii). The solvability of G is clear. Since the class of $\mathcal{PT}_c$-groups is a subclass of $\mathcal{PST}_c$-groups then each G_i is a solvable $\mathcal{PST}_c$-group for all i. Consequently, the assumption in (i) and Theorem 2.4.2 imply that G is a solvable $\mathcal{PST}_c$-group. Also, by hypothesis each $P_i \in \mathrm{Syl}_p(G_i)$ is an Iwasawa group for all primes p dividing the order of G_i. In particular, P is a direct product of Iwasawa groups. If P is abelian then P is an Iwasawa. If Q_8 is contained in P then (ii) satisfies the hypothesis of Theorem 2.2.3 (a), i.e. P is an Iwasawa. If Q_8 is not contained in P and P is not abelian then (iii) and Theorem 2.2.8 imply that P is an Iwasawa. This means that all Sylow subgroups of G are Iwasawa and since G is a $\mathcal{PST}_c$-group then by Lemma 2.5.1, G is a $\mathcal{PT}_c$-group. This concludes the proof. $\square$

Notice that in Theorem 2.3.2 we had a biconditional statement whereas in Corollary 2.5.2 if we suppose that $G := G_1 \times \cdots \times G_n$ is a $\mathcal{PT}_c$-group then we do not know if $P := P_1 \times \cdots \times P_n \in \mathrm{Syl}_p(G)$ is an Iwasawa group since we only supposed that each P_i was an Iwasawa-group.

Remark 2.5.3. Let $G := G_1 \times \cdots \times G_n$ be a finite group. Suppose that for all primes p dividing the order of G every Sylow p-subgroup of G is an Iwasawa group. Then G is a solvable $\mathcal{PT}_c$-group if and only if $G_1, \ldots, G_n$ are solvable $\mathcal{PT}_c$-groups such that $(|\gamma_*(G_i)|, |\mathrm{Fit}(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$.

Proof. Suppose that G is a solvable $\mathcal{PT}_c$-group. Then each G_i is a solvable $\mathcal{PT}_c$-group for all $i \in \{1, \ldots, n\}$. Also, $G \in \mathcal{PT}_c$ implies that $G \in \mathcal{PST}_c$. Hence by Theorem 2.4.2 we get that $(|\gamma_*(G_i)|, |\mathrm{Fit}(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$. Conversely, suppose that $G_1, \ldots, G_n$ are solvable $\mathcal{PT}_c$-groups such that $(|\gamma_*(G_i)|, |\mathrm{Fit}(G_j)|) = 1$

for all $i \neq j \in \{1, \ldots, n\}$. Then by Theorem 2.4.2 G is a solvable $\mathcal{PST}_c$-group. But now, hypothesis and Lemma 2.5.1 imply that G is a $\mathcal{PT}_c$-group. $\square$

Similar extensions can be made for solvable $\mathcal{T}_c$-groups.

Corollary 2.5.4. *Let $G := G_1 \times \cdots \times G_n$ be a finite group and $P := P_1 \times \cdots \times P_n \in \mathrm{Syl}_p(G)$ where $P_i \in \mathrm{Syl}_p(G_i)$ for some prime p dividing the order of G and $n \geq 2$. Suppose every Sylow subgroup of G_i is Dedekind, for all $i \in \{1, \ldots, n\}$.*
Then G is a solvable $\mathcal{T}_c$-group if the following hold:

(i) *$G_1, \ldots, G_n$ are solvable $\mathcal{T}_c$-groups such that $(|\gamma_*(G_i)|, |\mathrm{Fit}(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$.*

(ii) *if Q_8 is contained in P, then it is a subgroup of exactly one P_i and $P \backslash P_i$ is elementary 2-abelian or trivial; otherwise P is abelian.*

Proof. Suppose (i) and (ii). The solvability of G is clear. Since the class of $\mathcal{T}_c$-groups is a subclass of $\mathcal{PST}_c$-groups then each G_i is a solvable $\mathcal{PST}_c$-group for all i. Consequently, the assumption in (i) and Theorem 2.4.2 imply that G is a solvable $\mathcal{PST}_c$-group. Also, by hypothesis each $P_i \in \mathrm{Syl}_p(G_i)$ is a Dedekind group for all primes p dividing the order of G_i. In particular P is a direct product of Dedekind groups. If p is an odd prime, then each P_i is abelian, hence P is abelian. If $p = 2$ then (ii) implies that either P is abelian or a direct product of Q_8 and an elementary abelian 2-group. Hence, all Sylow subgroups of G are Dedekind and since G is a $\mathcal{PST}_c$-group then by Lemma 2.5.1, G is a $\mathcal{T}_c$-group. This concludes the proof. $\square$

Remark 2.5.5. Let $G := G_1 \times \cdots \times G_n$ be a finite group. Suppose that for all primes p dividing the order of G every Sylow p-subgroup of G is a Dedekind group. Then G is a solvable $\mathcal{T}_c$-group if and only if $G_1, \ldots, G_n$ are solvable $\mathcal{T}_c$-groups such that $(|\gamma_*(G_i)|, |\mathrm{Fit}(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$.

Proof. Suppose that G is a solvable $\mathcal{T}_c$-group. Then each G_i is a solvable $\mathcal{T}_c$-group for all $i \in \{1, \ldots, n\}$. Also, $G \in \mathcal{T}_c$ implies that $G \in \mathcal{PST}_c$. Hence by Theorem 2.4.2 we get that $(|\gamma_*(G_i)|, |\mathrm{Fit}(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$. Conversely, suppose that $G_1, \ldots, G_n$ are solvable $\mathcal{T}_c$-groups such that $(|\gamma_*(G_i)|, |\mathrm{Fit}(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$. Then by Theorem 2.4.2 G is a solvable $\mathcal{PST}_c$-group. But now, hypothesis and Lemma 2.5.1 imply that G is a $\mathcal{T}_c$-group. $\square$

If G is a $\mathcal{PT}_c$ or $\mathcal{T}_c$ group then the Fitting subgroup of G is an Iwasawa group or a Dedekind group, by Lemma 1.7.2. Subgroups of the Iwasawa (Dedekind) groups inherit the same properties, thus Sylow subgroups of the Fitting subgroup of the $\mathcal{PT}_c$ ($\mathcal{T}_c$) groups are Iwasawa (Dedekind) groups.

Open Question. Restricting Theorems 2.3.2 and 2.3.3 to Sylow subgroups of the Fitting subgroup, what are the additional conditions (if any) needed in order to extend these theorems to solvable $\mathcal{PT}_c$ and $\mathcal{T}_c$-groups?

2.6 The Fitting subgroup and direct products

Recall that if orders of groups are relatively prime then a direct product of $\mathcal{PST}, \mathcal{PT}$ or $\mathcal{T}$ groups was again a $\mathcal{PST}, \mathcal{PT}$ or $\mathcal{T}$-group, 2.0.5 and [1]. Trivially, it works the same way for the classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$. On the other hand, if H is any subnormal cyclic subgroup of a group G then H is contained in the Fitting subgroup of G and hence the question arises whether an assumption can be made that involves only the orders of the Fitting subgroups.

Theorem 2.6.1. *Let G_1 and G_2 be finite groups so that $(|\mathrm{Fit}(G_1)|, |\mathrm{Fit}(G_2)|) = 1$.*

(i) *G_1 and G_2 are $\mathcal{PST}_c$-groups if and only if $G_1 \times G_2 \in \mathcal{PST}_c$.*

(ii) *G_1 and G_2 are $\mathcal{T}_c$-groups if and only if $G_1 \times G_2 \in \mathcal{T}_c$.*

Proof. First we note that normal subgroups of $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups inherit the same properties. In particular if $G_1 \times G_2$ is a $\mathcal{PST}_c$, $\mathcal{PT}_c$, or $\mathcal{T}_c$ group then each G_i is a $\mathcal{PST}_c$, $\mathcal{PT}_c$, or $\mathcal{T}_c$ group.
Let H be a subnormal cyclic subgroup of $G_1 \times G_2$. Then H is contained in the $\mathrm{Fit}(G_1 \times G_2)$. Since $(|\mathrm{Fit}(G_1)|, |\mathrm{Fit}(G_2)|) = 1$ and $\mathrm{Fit}(G_1 \times G_2) = \mathrm{Fit}(G_1) \times \mathrm{Fit}(G_2)$ then $H = H_1 \times H_2$ where each H_i is a subgroup of $\mathrm{Fit}(G_i)$. In particular H_i is subnormal subgroup of G_i.
Let $G_p = G_p^1 \times G_p^2 \in \mathrm{Syl}_p(G_1 \times G_2)$ where $G_p^i \in \mathrm{Syl}_p(G_i)$. If G_i is a $\mathcal{PST}_c$ then H_i is Sylow-permutable in G_i, i.e. H_i permutes with G_p^i. But G_j centralizes G_i, hence H_i permutes with G_p^j. Thus H and G_p permute, that is $G_1 \times G_2$ is $\mathcal{PST}_c$ group.
If G_i is a $\mathcal{T}_c$ group then H_i is normal subgroup in G_i, that is $H_1 \times H_2$ is normal subgroup of $G_1 \times G_2$. Hence, $G_1 \times G_2$ is $\mathcal{T}_c$ group. This proves (i) and (ii). □

To prove a similar result for $\mathcal{PT}_c$-groups, one has to be concerned about subgroups that cannot be written as a direct product, that is subgroups that are not of the form $K_1 \times K_2$, $K_i \in G_i$. In particular, make the same assumptions as in Theorem 2.6.1 and consider a cyclic subnormal subgroup H of the group $G_1 \times G_2$, where each $G_i \in \mathcal{PT}_c$.

Let H_p be a Sylow p-subgroup of H, for some prime dividing the order of H. Then H_p is contained in $O_p(G)$ and Fitting subgroups of G_1 and G_2 have relatively prime orders, so H_p will be contained in G_1 or G_2. Now, p'-elements of $G_1 \times G_2$ present no problem, but if $g_1 g_2$ is any p-element of $G_1 \times G_2$, where $g_i \in G_i$, then it is not clear whether H_p will permute with $\langle g_1 g_2 \rangle$, even though H_p permutes with each $\langle g_i \rangle$.

Remark 2.6.2. Let $g_1 g_2$ be any p-element of $G_1 \times G_2$ for some prime divisor p of the order of $G_1 \times G_2$. Then $\operatorname{ord}(g_1 g_2) = p^{\alpha}$ for some integer α. The reader should keep in mind that $g_1 g_2 = (g_1, g_2)$. Now, $\operatorname{ord}(g_1 g_2) = \operatorname{lcm}(\operatorname{ord}(g_1), \operatorname{ord}(g_2)) = p^{\alpha}$. This means that $\operatorname{ord}(g_1) | p^{\alpha}$ and $\operatorname{ord}(g_2) | p^{\alpha}$. Therefore, $\operatorname{ord}(g_i) = p^{\alpha_i}$, for some integer $\alpha_i \leq \alpha$. Thus, g_i is a p-element of G_i.

Lemma 2.6.3. *Let $G_1 \times G_2$ be a finite group and let $g_1 g_2 \in G_1 \times G_2$ be any p-element for some prime p dividing the order of $G_1 \times G_2$, where $g_i \in G_i$. Then, $g_1^m g_2 \in \langle g_1 g_2 \rangle$ if and only if for some integer n,*

$$m \equiv (n \cdot \operatorname{ord}(g_2) + 1) \bmod \operatorname{ord}(g_1),$$

where $\operatorname{ord}(g_i)$ *is the order of* g_i.

Proof. Let $G_1 \times G_2$ be a finite group and let $g_1 g_2 \in G_1 \times G_2$ be any p-element. Let $\operatorname{ord}(g_i) = p^{\alpha_i}$ for $i \in \{1, 2\}$. Suppose that $g_1^m g_2 \in \langle g_1 g_2 \rangle$. Then for some integer j, $g_1^m g_2 = (g_1 g_2)^j = g_1^j g_2^j$, that is

$$m \equiv j \bmod p^{\alpha_1} \quad \text{and} \quad 1 \equiv j \bmod p^{\alpha_2}.$$

Now, for some integers k_1 and k_2 we have,

$$j - m = k_1 \cdot p^{\alpha_1} \text{ and } j - 1 = k_2 \cdot p^{\alpha_2}$$

$$m + k_1 \cdot p^{\alpha_1} = 1 + k_2 \cdot p^{\alpha_2}$$

$$k_1 \cdot p^{\alpha_1} = (1 + k_2 \cdot p^{\alpha_2}) - m.$$

Therefore, $m \equiv (1 + k_2 \cdot p^{\alpha_2}) \bmod p^{\alpha_1}$.
Conversely, if $m \equiv (1 + n \cdot p^{\alpha_2}) \bmod p^{\alpha_1}$ for some integer n, then

$$g_1^m = g_1^{1+n \cdot p^{\alpha_2}}.$$

But, $g_2^{1+n \cdot p^{\alpha_2}} = g_2$, since the order of g_2 is p^{α_2}. Hence,

$$g_1^m g_2 = g_1^{1+n \cdot p^{\alpha_2}} g_2^{1+n \cdot p^{\alpha_2}} = (g_1 g_2)^{1+n \cdot p^{\alpha_2}} \in \langle g_1 g_2 \rangle.$$

□

Theorem 2.6.4. *Let G_1 and G_2 be finite groups so that $(|\mathrm{Fit}(G_1)|, |\mathrm{Fit}(G_2)|) = 1$. Let H_p be a subnormal cyclic p-subgroup of G_i and g_1g_2 be any p-element of $G_1 \times G_2$, where $g_i \in G_i$. Then, $G_1 \times G_2 \in \mathcal{PT}_c$ if and only if each G_i is a $\mathcal{PT}_c$ -group so that*

$$h^{g_i} = g_i^{n\cdot \mathrm{ord}(g_j)} h^k$$

for all $h \in H_p$ and every prime divisor p of the order of G_i and some integers n, k and $i \neq j \in \{1, 2\}$.

Proof. First we note that normal subgroups of $\mathcal{PT}_c$-groups inherit the same properties. In particular if $G_1 \times G_2$ is a $\mathcal{PT}_c$-group then each G_i is a $\mathcal{PT}_c$-group. Without loss of generality, let H_p be any subnormal cyclic p-subgroup of G_1 and g_1g_2 a p-element of $G_1 \times G_2$, where $g_i \in G_i$. Let $\mathrm{ord}(g_i) = p^{\alpha_i}$ for $i \in \{1, 2\}$. Now, H_p is permutable subgroup of G_1 and of $G_1 \times G_2$. Thus, H_p permutes with $\langle g_1 \rangle$ and with $\langle g_1g_2 \rangle$. In particular, $hg_1 = g_1^m h^k$, where $h, h^k \in H_p$ and $g_1, g_1^m \in \langle g_1 \rangle$, for some integers m and k. Now, consider $h(g_1g_2) \in H_p\langle g_1g_2 \rangle$. Then

$$h(g_1g_2) = hg_1g_2 = g_1^m h^k g_2 = (g_1^m g_2)h^k.$$

Since H_p and $\langle g_1g_2 \rangle$ permute then $g_1^m g_2 \in \langle g_1g_2 \rangle$. Now, Lemma 2.6.3 implies that $m \equiv (1 + n \cdot p^{\alpha_2}) \bmod p^{\alpha_1}$, for some integer n. This means that $g_1^m = g_1^{1+n\cdot p^{\alpha_2}}$. Hence,

$$hg_1 = g_1^m h^k = g_1^{1+n\cdot p^{\alpha_2}} h^k = g_1 g_1^{n\cdot p^{\alpha_2}} h^k$$
$$g_1^{-1} h g_1 = g_1^{n\cdot p^{\alpha_2}} h^k.$$

Conversely, suppose that G_i is a $\mathcal{PT}_c$ group and let H be a subnormal cyclic subgroup of $G_1 \times G_2$. Then H is contained in the $\mathrm{Fit}(G_1 \times G_2)$. Let H_p be a direct factor of H for some prime divisor p of the order of H. Since $(|\mathrm{Fit}(G_1)|, |\mathrm{Fit}(G_2)|) = 1$ then H_p is contained in $F_p^1 \in \mathrm{Syl}_p(\mathrm{Fit}(G_1))$ or $F_p^2 \in \mathrm{Syl}_p(\mathrm{Fit}(G_2))$. Without loss of generality suppose that $H_p \in F_p^1$. Since $\mathcal{PT}_c$ is a subclass of $\mathcal{PST}_c$ then $G_1 \times G_2$ is a $\mathcal{PST}_c$ by part (i) of Theorem 2.6.1. Let $g \in G_1 \times G_2$ and consider a cyclic group generated by g. Then $\langle g \rangle = \langle g_p \rangle \times \langle g_{p'} \rangle$ where g_p is a p-element and $g_{p'}$ is a p'-element of $G_1 \times G_2$. Since $G_1 \times G_2$ is a $\mathcal{PST}_c$ group and H_p is subnormal in G_1, i.e. subnormal subgroup of $G_1 \times G_2$, then p'-elements of $G_1 \times G_2$ induce power automorphism in H_p. That is, H_p permutes with $\langle g_{p'} \rangle$.

Now let $g_p = g_1g_2$ where $g_i \in G_i$ and let $\mathrm{ord}(g_i) = p^{\alpha_i}$ for $i \in \{1, 2\}$. Since G_1 is a $\mathcal{PT}_c$ then H_p and $\langle g_1 \rangle$ permute. In particular, $hg_1 = g_1^m h^k$, where $h, h^k \in H_p$ and $g_1, g_1^m \in \langle g_1 \rangle$, for some integers m and k. But then we have that $h(g_1g_2) = hg_1g_2 = g_1^m h^k g_2 = (g_1^m g_2)h^k$. By hypothesis, $h^{g_1} = g_1^{n\cdot p^{\alpha_2}} h^k$, i.e.

$$hg_1 = g_1^m h^k = g_1^{1+n\cdot p^{\alpha_2}} h^k.$$

Now, Lemma 2.6.3 implies that $g_1^m g_2 \in \langle g_1 g_2 \rangle$. Hence, H_p and $\langle g_p \rangle$ permute, that is H_p and $\langle g \rangle$ permute for all $g \in G_1 \times G_2$. Since every p-factor of $H = H_1 \times H_2$ is permutable in $G_1 \times G_2$ then we conclude that H is permutable in $G_1 \times G_2$ and $G_1 \times G_2$ is a $\mathcal{PT}_c$ group. □

From Theorem 2.6.4 we immediately get the following result.

Lemma 2.6.5. *Let $G_1 \times G_2$ be a finite $\mathcal{PT}_c$ group, H_p be a subnormal cyclic p-subgroup of G_1 and $g_1 g_2$ be any p-element of $G_1 \times G_2$, where $g_i \in G_i$.*
If $\operatorname{ord}(g_1) \leq \operatorname{Exp}(P_2)$ *for any Sylow p-subgroup of G_2, then elements of $\langle g_1 \rangle$ induce power automorphisms in H_p.*

Proof. Let $\operatorname{ord}(g_i) = p^{\alpha_i}$ for $i \in \{1, 2\}$. Since $G_1 \times G_2$ is a $\mathcal{PT}_c$-group and each G_i is normal in $G_1 \times G_2$ then G_i is a $\mathcal{PT}_c$-group. Hence, H_p is permutable subgroup of G_1 and of $G_1 \times G_2$. Thus, H_p permutes with $\langle g_1 \rangle$ and with $\langle g_1 g_2 \rangle$. In particular, $hg_1 = g_1^m h^k$, where $h, h^k \in H_p$ and $g_1, g_1^m \in \langle g_1 \rangle$, for some integers m and k. Now, consider $h(g_1 g_2) \in H_p \langle g_1 g_2 \rangle$. Then

$$h(g_1 g_2) = hg_1 g_2 = g_1^m h^k g_2 = (g_1^m g_2) h^k.$$

Since H_p and $\langle g_1 g_2 \rangle$ permute then $g_1^m g_2 \in \langle g_1 g_2 \rangle$. Now, Lemma 2.6.3 implies that $m \equiv (1 + n \cdot p^{\alpha_2}) \bmod p^{\alpha_1}$. This means that $g_1^m = g_1^{1+n \cdot p^{\alpha_2}}$. But $\operatorname{ord}(g_1) \leq Exp(P_2)$, which implies that $g_1^{1+n \cdot p^{\alpha_2}} = g_1$. Now, $hg_1 = g_1^m h^k = g_1 h^k$, i.e. $h^{g_1} = h^k$. □

In order to find necessary and sufficient conditions for a direct product of solvable $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups to stay in the class further analysis and a better understanding of these classes of groups is required.
It would be interesting to examine which subgroups inherit the properties of the group and identify the structure of the Sylow subgroups. A lot of examples suggest that Sylow subgroups of solvable $\mathcal{PT}_c$ and $\mathcal{T}_c$-groups are Iwasawa or Dedekind groups, respectively. Nonetheless, more examples of solvable $\mathcal{PT}_c$ or a $\mathcal{T}_c$ groups that contain a non Iwasawa or a non Dedekind Sylow subgroups must be constructed and investigated.

2.7 Examples

This section provides examples of direct products of $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups. To build a solvable $\mathcal{T}_c$ group we use Robinson's [12] construction as described on page 176. Then we use Theorem 2.4.2 and Lemma 2.5.1 to build a $\mathcal{PST}_c$ group that is not a $\mathcal{PT}_c$ and $\mathcal{PT}_c$ group that is not a $\mathcal{T}_c$. This illustrates that classes of $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups are proper subclasses of each other. Also, we provide an example of a non $\mathcal{PST}_c$ group.

We hope that examples will help reader to see the fundamental differences between the classes of solvable $\mathcal{T}, \mathcal{PT}, \mathcal{PST}$ groups and the classes of solvable $\mathcal{T}_c, \mathcal{PT}_c, \mathcal{PST}_c$ groups. In particular, notice that the nilpotent residual is no longer a Hall-subgroup of the group G and that subgroup closure fails as well.

Example 2.7.1.

$$G = \left\langle a, b, c, d, f \,\middle|\, \begin{array}{l} a^7 = b^3 = c^2 = d^3 = f^5 = (ac)^2 = (bc)^2 = a^5 a^d = 1 \\ [a, b] = [c, d] = [a, f] = [b, f] = [c, f] = [d, f] = [b, d] = 1 \end{array} \right\rangle$$

(i) $G \cong ((C_7 \times C_3) \rtimes (C_2 \times C_3)) \times C_5$.

(ii) The Fitting subgroup of G is $\mathrm{Fit}(G) = (C_7 \times C_3) \times C_5$.

(iii) The nilpotent residual of G is $\gamma_*(G) = C_7 \times C_3$.

(iv) $\gamma_*(G)$ is a Hall-subgroup of the $\mathrm{Fit}(G)$ but not a Hall-subgroup of G.

(v) G is a solvable $\mathcal{T}_c$-group that is not a $\mathcal{T}$-group.

(vi) Let $H = \langle b, c, d\rangle$ be a subgroup of G. $H \cong S_3 \times C_3$. The nilpotent residual $\gamma_*(H) = \langle b\rangle$ is not a Hall subgroup of the $\mathrm{Fit}(H) = \langle b, d\rangle$. Thus, H is not a $\mathcal{T}_c$ group.

1. Consider $G \times D_8$, where $D_8 = \langle x, y | x^4 = y^2 = 1, x^y = x^{-1}\rangle$

   (i) D_8 is a $\mathcal{PST}_c$-group and $(|\mathrm{Fit}(G)|, |\gamma_*(D_8)|) = (|\gamma_*(G)|, |\mathrm{Fit}(D_8)|) = 1$. Therefore, $G \times D_8$ is a $\mathcal{PST}_c$-group.

   (ii) $G \times D_8$ is not a $\mathcal{PT}_c$-group since the cyclic subnormal subgroup $\langle y\rangle$ does not permute with $\langle xy\rangle$.

2. Consider $G \times M$, where $M = \langle x, y | x^8 = y^2 = 1, x^y = x^5\rangle$

   (i) M is a $\mathcal{PST}_c$-group and $(|\mathrm{Fit}(G)|, |\gamma_*(M)|) = (|\gamma_*(G)|, |\mathrm{Fit}(M)|) = 1$, hence $G \times M$ is a $\mathcal{PST}_c$-group.

   (ii) Since every Sylow subgroup of $G \times M$ is Iwasawa then $G \times M$ is a $\mathcal{PT}_c$-group, but $G \times M$ is not a $\mathcal{T}_c$-group since the cyclic subnormal subgroup $\langle y\rangle$ is not normal.

3. Consider $G \times Q_8$

   (i) Q_8 is a $\mathcal{PST}_c$-group and $(|\mathrm{Fit}(G)|, |\gamma_*(Q_8)|) = (|\gamma_*(G)|, |\mathrm{Fit}(Q_8)|) = 1$, hence $G_1 \times Q_8$ is a $\mathcal{PST}_c$-group.

   (ii) Since every Sylow subgroup of $G \times Q_8$ is Dedekind then $G \times Q_8$ is a $\mathcal{T}_c$-group.

4. Let G_1 be a dihedral group of order 12. We will consider the following presentation of this group: $G_1 := \langle x, y \mid x^6 = y^2 = 1, x^y = x^{-1} \rangle$. Clearly, $G_1 \cong (C_3 \rtimes C_2) \times C_2$ and it is a solvable $\mathcal{T}_c$-group.
   Now, consider $G \times G_1$.

   (i) G and G_1 are each solvable $\mathcal{T}_c$-groups with
   $\gamma_*(G \times G_1) = (C_7 \times C_3) \times C_3$ being a Hall subgroup of the $\mathrm{Fit}(G \times G_1) = ((C_7 \times C_3) \times C_5) \times (C_3 \times C_1)$.

   (ii) But the order of the $\mathrm{Fit}(G_1) = C_3 \times C_2$ and the order of $\gamma_*(G) = C_7 \times C_3$ are not relatively prime, therefore $G \times G_1$ is not a $\mathcal{PST}_c$-group.

Chapter 3 The Intersection map of subgroups

In this chapter we analyze the behavior of a collection of cyclic normal, permutable and Sylow-permutable subgroups under the intersection map into a fixed subgroup of a group. In particular, we tie the concept of normal, permutable and Sylow-permutable cyclic sensitivity with that of $\mathcal{T}_c$, $\mathcal{PT}_c$ and $\mathcal{PST}_c$ groups. In the process we provide another way of looking at Dedekind, Iwasawa and nilpotent groups. The intersection map of subgroups in connection to the classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$ is a collaborative work with my advisor James Beidleman.

3.1 Background

Definition 3.1.1. A subgroup H of the group G is said to be *normal sensitive* if whenever X is a normal subgroup of H there is a normal subgroup Y of G such that $X = Y \cap H$, that is if the map $Y \mapsto H \cap Y$ sends the lattice of normal subgroups of G onto the lattice of normal subgroups of H.

Permutable sensitive (*S-permutable sensitive*) are defined in the similar fashion by requiring X and Y to be permutable (S-permutable) subgroups of H and G respectively.

Recall, that a permutable (S-permutable) subgroup of the group G is subnormal. We would like to note, that the set of all permutable subgroups need not be a sublattice of the lattice of subnormal subgroups of a group G. Hence, in the case of permutability the intersection map is not necessarily a lattice map. An example that addresses the latter can be found in [6] on page 220, Example 1. On the other hand, the collection of S-permutable subgroups is a sublattice of the lattice of subnormal subgroups of G. For details reader may consult [10] and [14].

Several authors, Bauman [4], Beidleman and Ragland [6], have tied the concept of normal, permutable and S-permutable sensitivity with $\mathcal{T}$, $\mathcal{PT}$ and $\mathcal{PST}$ groups.

Theorem 3.1.2. *Let G be a finite group.*

(1) (Beidleman, Ragland [6]) *G is a solvable $\mathcal{PST}$ ($\mathcal{PT}$)-group if and only if every subgroup of G is S-permutable (permutable) sensitive in G.*

(2) (Bauman [4]) *G is a solvable $\mathcal{T}$-group if and only if every subgroup of G is normal sensitive in G.*

Ragland and Beidleman in [6] go on further by asking a question whether one can restrict S-permutable, permutable, and normal sensitivity to normal subgroups and deduce that G is still a $\mathcal{PST}$, $\mathcal{PT}$, or a $\mathcal{T}$ group respectively. While they have affirmatively answered the question about $\mathcal{PST}$ and $\mathcal{T}$ groups in [6], the question about permutable sensitivity restricted to normal subgroups was answered later in [3].

Theorem 3.1.3. *Let G be a group.*

(1) (Beidleman, Ragland [6]) *G is a $\mathcal{PST}$ ($\mathcal{T}$)-group if and only if every normal subgroup of G is S-permutable (normal) sensitive in G.*

(2) (Ballester-Bolinches, A., Beidleman, J.C., Cossey, J., Esteban-Romero, R., Ragland, M.F., Schmidt, J. [3]) *G is a solvable $\mathcal{PT}$-group if and only if every normal subgroup of G is permutable sensitive in G.*

Our interest lies in developing similar connections as in Theorems 3.1.2 and 3.1.3 with classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$. In particular, if we restrict the intersection map to cyclic subgroups, then what can we say about the behavior of a collection of cyclic normal, permutable and S-permutable subgroups under this restricted intersection map.

3.2 The intersection map of subgroups and the classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$.

Definition 3.2.1. Let H be a subgroup of the group G.

(1) H is *normal cyclic sensitive* if whenever X is a normal cyclic subgroup of H there is a normal cyclic subgroup Y of G such that $X = Y \cap H$.

(2) H is *permutable cyclic sensitive* if whenever X is a permutable cyclic subgroup of H there is a permutable cyclic subgroup Y of G such that $X = Y \cap H$.

(3) H is *S-permutable cyclic sensitive* if whenever X is a S-permutable cyclic subgroup of H there is a S-permutable cyclic subgroup Y of G such that $X = Y \cap H$.

Definition 3.2.1 is an analog of the normal, permutable and S-permutable sensitivity, but the following two lemmas will provide a much simpler and more natural way of looking at the cyclic sensitivity.

Lemma 3.2.2. (R. Schmidt [15], Lemma 5.2.11 page 224) *If M is a cyclic permutable subgroup of the group G, then every subgroup of M is permutable in G.*

Lemma 3.2.3. (P.Schmid [14]) *Let X be a cyclic subgroup of a finite group G and let Y be a subgroup of X. If X is S-permutable in G, then Y is also S-permutable in G.*

Let H be normal cyclic sensitive subgroup of G and X normal cyclic subgroup of H. Then there exist a normal cyclic subgroup Y of G such that $X = Y \cap H$. But now X is the unique cyclic subgroup of Y, i.e X is characteristic in Y and Y is normal subgroup of G. Thus, X is normal subgroup of G.
Similarly, if H is permutable (S-permutable) cyclic sensitive then and X is permutable (S-permutable) cyclic subgroup of H then there is a permutable (S-permutable) cyclic subgroup Y of G such that $X = H \cap Y$. Then by Lemmas 3.2.2 (3.2.3) X is permutable (S-permutable) subgroup of G. Hence, normal, permutable and S-permutable cyclic sensitivity is equivalent to the following.

Remark 3.2.4. Let G be a finite group and H a subgroup of G.

1. H is normal cyclic sensitive in G if every normal cyclic subgroup of H is normal subgroup of G.

2. H is permutable cyclic sensitive in G if every permutable cyclic subgroup of H is permutable subgroup of G.

3. H is S-permutable cyclic sensitive in G if every S-permutable cyclic subgroup of H is S-permutable subgroup of G.

From now on, we will use Remark 3.2.4 in place of Definition 3.2.1.
In general, normal subgroups of normal, permutable, S-permutable cyclic sensitive groups do not inherit the same properties. That is, if H is normal cyclic sensitive in G and K is normal subgroup of H then K need not be normal cyclic sensitive as the following example shows.

Example 3.2.5. Consider the following group

$$G = \langle x, y, z \mid x^6 = y^2 = z^3 = (xy)^2 = [x, z] = [y, z] = 1 \rangle$$

$G \cong D_{12} \times C_3$.Let $H = \langle x^2, y, z \rangle$ and $K = \langle x^2, z \rangle$. H is normal cyclic sensitive in G since the only normal cyclic subgroups of H are $\langle x^2 \rangle$ and $\langle z \rangle$ and both of them are normal in G. Now, K is normal subgroup of H but K has a normal cyclic subgroup $\langle x^2 z \rangle$ that is not normal in H nor G. Thus, K is not normal cyclic sensitive.

Recall that Theorem 3.1.2 by Beidleman, Ragland [6] and Bauman [4] relates solvable $\mathcal{PST}$, $\mathcal{PT}$ and $\mathcal{T}$ groups to S-permutable, permutable and normal sensitivity. Our first result addresses a similar question in connection to cyclic sensitivity.

Theorem 3.2.6. *Let G be a finite group.*

(1) *G is a nilpotent group if and only if every subgroup of G is S-permutable cyclic sensitive.*

(2) *G is an Iwasawa group if and only if every subgroup of G is permutable cyclic sensitive.*

(3) *G is a Dedekind group if and only if every subgroup of G is normal cyclic sensitive.*

Proof. (1) If G is nilpotent then every subgroup of G is subnormal and every Sylow subgroup of G is normal. Hence, every subgroup is S-permutable cyclic sensitive.
Conversely, suppose that every subgroup of G is S-permutable cyclic sensitive. Let X be any cyclic subgroup of G. Then X is S-permutable cyclic sensitive, i.e. X is S-permutable in G. Kegel [10] has shown that S-permutable subgroups are subnormal, then X is subnormal subgroup of G. Hence, we conclude that every subgroup of G is subnormal, in particular G is nilpotent.
(2) If G is an Iwasawa group then every subgroup is permutable in G and consequently permutable cyclic sensitive.
Conversely, let X be any cyclic subgroup of G, then X is permutable cyclic sensitive. Thus, X is permutable subgroup of G, and G is an Iwasawa group.
(3) If G is a Dedekind group then every subgroup is normal cyclic sensitive.
Conversely, if X is any cyclic subgroup of G then X is normal cyclic sensitive and hence normal in G. Thus, G is a Dedekind group. □

Theorem 3.2.7. *Let G be a finite group.*

(1) *G is a $\mathcal{PST}_c$-group if and only if every subnormal subgroup of G is S-permutable cyclic sensitive.*

(2) *G is a $\mathcal{PT}_c$-group if and only if every subnormal subgroup of G is permutable cyclic sensitive.*

(3) *G is a $\mathcal{T}_c$-group if and only if every subnormal subgroup of G is normal cyclic sensitive.*

Proof. (1) Suppose that G is a $\mathcal{PST}_c$-group. Let H be any subnormal subgroup of G and X an S-permutable cyclic subgroup of H. Then X is subnormal subgroup of G and hence S-permutable. Conversely, if X is subnormal cyclic subgroup of G then hypothesis implies that X is S-permutable.
(2) Suppose that G is a $\mathcal{PT}_c$-group. Let H be any subnormal subgroup of G and X a permutable cyclic subgroup of H. Then X is subnormal subgroup of G and hence permutable. Conversely, if X is subnormal cyclic subgroup of G then hypothesis implies that X is permutable.
(3) Suppose that G is a $\mathcal{T}_c$-group. Let H be any subnormal subgroup of G and X a normal cyclic subgroup of H. Then X is subnormal subgroup of G and hence normal in G. Conversely, if X is subnormal cyclic subgroup of G then hypothesis implies that X is normal in G. □

One would hope that it is possible to replace "subnormal" with "normal" in Theorem 3.2.7 so that one can have an analog to Theorem 3.1.3, but it is not the case as the following example shows.

Example 3.2.8. Consider the dihedral group of order 16.

$$D_{16} = \langle x, y, \mid x^8 = y^2 = (xy)^2 = 1 \rangle$$

One can check that normal subgroups of D_{16} are normal cyclic sensitive but D_{16} is not a $\mathcal{T}_c$-group since $\langle y \rangle$ is subnormal cyclic subgroup of D_{16} which is not normal.

Also note, that if H is any subgroup of G such that every subgroup of H is normal cyclic sensitive in H then in general H is not normal cyclic sensitive in G.

Example 3.2.9. Consider again a dihedral group of order 16 with the same generators as in the previous example. Let $H = \langle x^4, y \rangle$. H is isomorphic to K_4 and hence every subgroup of H is normal cyclic sensitive in H, but H is not normal cyclic sensitive in G since $\langle y \rangle$ is subnormal cyclic subgroup of D_{16} which is not normal.

Robinson in [12] has proved that if every subgroup of a group G is $\mathcal{PST}_c$ then G is a solvable $\mathcal{PST}$ group. Same is true for solvable $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups. Robinson's results and Theorem 3.2.7 motivate the following theorem.

Theorem 3.2.10. *Let G be a finite group.*

(1) *G is a solvable $\mathcal{PST}$-group if and only if every subnormal subgroup of H is S-permutable cyclic sensitive in H for all subgroups H of G.*

(2) *G is a solvable $\mathcal{PT}$-group if and only if every subnormal subgroup of H is permutable cyclic sensitive in H for all subgroups H of G.*

(3) *G is a solvable $\mathcal{T}$-group if and only if every subnormal subgroup of H is normal cyclic sensitive in H for all subgroups H of G.*

Proof. (1) Suppose G is a solvable $\mathcal{PST}$-group and let K be any subnormal subgroup of H. Let X be an S-permutable cyclic subgroup of K. Since S-permutable subgroups are subnormal 1.5.3 part (iv), then X is a subnormal subgroup of H. Since solvable $\mathcal{PST}$ groups are subgroup closed, then H is a solvable $\mathcal{PST}$ group. Thus, X is S-permutable in H, i.e. every subnormal subgroup of H is S-permutable cyclic sensitive in H.
Conversely, if every subnormal subgroup of H is S-permutable cyclic sensitive in H, then by Theorem 3.2.7 H is a $\mathcal{PST}_c$-group. Now, every subgroup of G is a $\mathcal{PST}_c$ group, thus Theorem 2.5 in [12] implies that G is a solvable $\mathcal{PST}$ group.
(2) Suppose G is a solvable $\mathcal{PT}$-group and let K be any subnormal subgroup of H. Let X be a permutable cyclic subgroup of K. Since permutable subgroups are subnormal 1.5.3 part (ii), then X is a subnormal subgroup of H. Since solvable $\mathcal{PT}$ groups are subgroup closed, then H is a solvable $\mathcal{PT}$ group. Thus, X is permutable in H, i.e. every subnormal subgroup of H is permutable cyclic sensitive in H.
Conversely, if every subnormal subgroup of H is permutable cyclic sensitive in H, then by Theorem 3.2.7 H is a $\mathcal{PT}_c$-group. Now, every subgroup of G is a $\mathcal{PT}_c$ group, thus Theorem 2.5 in [12] implies that G is a solvable $\mathcal{PT}$ group.
(3) Suppose G is a solvable $\mathcal{T}$-group and let K be any subnormal subgroup of H. Let X be a normal cyclic subgroup of K. Then X is a subnormal subgroup of H. Since solvable $\mathcal{T}$ groups are subgroup closed, then H is a solvable $\mathcal{T}$ group. Thus, X is normal in H, i.e. every subnormal subgroup of H is normal cyclic sensitive in H.
Conversely, if every subnormal subgroup of H is normal cyclic sensitive in H, then by Theorem 3.2.7 H is a $\mathcal{T}_c$-group. Now, every subgroup of G is a $\mathcal{PT}_c$ group, thus Theorem 2.5 in [12] implies that G is a solvable $\mathcal{T}$ group. □

3.3 Sylow Subgroups and the Intersection Map

We switch our attention to Sylow subgroups of a group G. Since cyclic subgroups can be written as a direct product of cyclic p-groups of relatively prime orders, it is natural to look at the Sylow subgroups.

Theorem 3.3.1. *Let G be a finite group and let G_p be a Sylow p-subgroup of G.*

(1) *G is a Dedekind group if and only if every subgroup of G_p is normal cyclic sensitive in G for all primes p dividing the order of G.*

(2) *G is an Iwasawa group if and only if every subgroup of G_p is permutable cyclic sensitive in G for all primes p dividing the order of G.*

(3) *G is nilpotent group if and only if G_p is S-permutable cyclic sensitive in G for all primes p dividing the order of G.*

Proof. (1) If G is a Dedekind group then every subgroup is normal and hence every subgroup of a Sylow subgroup G_p is normal cyclic sensitive.
Conversely, suppose that every subgroup of G_p is normal cyclic sensitive in G for all primes p dividing the order of G. First, we would like to note, that hypothesis implies that G_p is normal cyclic sensitive in G since if X is any normal cyclic subgroup of G_p then it is normal cyclic sensitive in G, that is X is normal subgroup of G. Second, if H is any subgroup of G_p and X is normal cyclic subgroup of H then hypothesis implies that X is normal in G and consequently normal in G_p. This means that every subgroup of G_p is normal cyclic sensitive in G_p. Now, Theorem 3.2.6 implies that G_p is a Dedekind group. Hence, if X is any cyclic subgroup of G and X_p is a Sylow p-subgroup of X, for some prime p dividing the order of X, then X_p is normal cyclic subgroup of G_p and G_p is normal cyclic sensitive in G implies X_p is normal in G. Since X is a direct product of its Sylow subgroups, then X is normal in G. Thus, G is a Dedekind group.
(2) If G is an Iwasawa group then every subgroup is permutable in G and consequently permutable cyclic sensitive. That is every subgroup of a Sylow subgroup G_p is permutable cyclic sensitive.
Conversely, suppose that every subgroup of G_p is permutable cyclic sensitive in G for all primes p dividing the order of G. Note that hypothesis implies that G_p is permutable cyclic sensitive in G since if X is any permutable cyclic subgroup of G_p then it is permutable cyclic sensitive in G, that is X is permutable subgroup of G. Second, if H is any subgroup of G_p and X is permutable cyclic subgroup of H then hypothesis

implies that X is permutable in G and consequently permutable in G_p. This means that every subgroup of G_p is permutable cyclic sensitive in G_p. Now, Theorem 3.2.6 implies that G_p is an Iwasawa group. Hence, if X is any cyclic subgroup of G and X_p is a Sylow p-subgroup of X, for some prime p dividing the order of X, then X_p is permutable cyclic subgroup of G_p and G_p is permutable cyclic sensitive in G implies X_p is permutable in G. Since X is a direct product of its Sylow subgroups, then X is permutable in G. Thus, G is an Iwasawa group.
(3) Suppose G is nilpotent group, then every subgroup of G is S-permutable in G. Thus, every Sylow subgroup is S-permutable cyclic sensitive.
Conversely, suppose that every Sylow subgroup of G is S-permutable cyclic sensitive. Let H be any cyclic subgroup of G and let H_p be a Sylow p-subgroup of H, for some prime p dividing the order of H. Now, H_p is contained in G_p. Since G_p is S-permutable cyclic sensitive then H_p is S-permutable in G. But, Kegel [10] has shown that S-permutable subgroups are subnormal, hence H_p is subnormal subgroup of G. Thus, we conclude H is subnormal subgroup of G. Since, every cyclic subgroup of G is subnormal in G then every subgroup is subnormal in G, that is G is nilpotent. □

Note, that in Theorem 3.3.1 parts (1) and (2) can be restated as follows.

Corollary 3.3.2. *Let G be a finite group and G_p be a Sylow p-subgroup of G.*

(1) *G is a Dedekind group if and only if G_p is a Dedekind group and normal cyclic sensitive in G for all primes p dividing the order of G.*

(2) *G is an Iwasawa group if and only if G_p is an Iwasawa group and permutable cyclic sensitive in G for all primes p dividing the order of G.*

Proof. (1) If G is a Dedekind group then every subgroup of G is a Dedekind group and normal in G. In particular, G_p is a Dedekind group and normal cyclic sensitive in G for all primes p dividing the order of G. Conversely, suppose that G_p is a Dedekind group and normal cyclic sensitive in G for all primes p dividing the order of G. Then, if K is any subgroup of G_p and X is normal cyclic subgroup of K then X is normal cyclic subgroup of G_p and hence normal subgroup of G. That is, every subgroup of G_p is normal cyclic sensitive in G. Hence, by theorem 3.3.1 G is a Dedekind group.
(2) If G is an Iwasawa group then every subgroup of G is an Iwasawa group and permutable in G. In particular, G_p is an Iwasawa group and permutable cyclic sensitive in G for all primes p dividing the order of G. Conversely, G_p is an Iwasawa group and permutable cyclic sensitive in G for all primes p dividing the order of G. So, if K is any subgroup of G_p and X is a permutable cyclic subgroup of K then X is a

permutable cyclic subgroup of G_p and hence a permutable subgroup of G. That is, every subgroup of G_p is permutable cyclic sensitive in G. Hence, by Theorem 3.3.1 G is an Iwasawa group. □

If X is any subnormal cyclic subgroup of a group G then X is contained in the Fitting subgroup of G and in particular the Sylow p-subgroup X_p of X, for some prime dividing the order of X, lies in the Sylow p-subgroup of the Fitting, that is X_p is contained in $O_p(G)$.

Theorem 3.3.3. *Let G be a finite group.*

(1) *G is a $\mathcal{T}_c$ group if and only if every subgroup of $O_p(G)$ is normal cyclic sensitive in G for all primes p dividing the order of G.*

(2) *G is a $\mathcal{PT}_c$ group if and only if every subgroup of $O_p(G)$ is permutable cyclic sensitive in G for all primes p dividing the order of G.*

(3) *G is a $\mathcal{PST}_c$-group if and only if $O_p(G)$ is S-permutable cyclic sensitive in G for all primes p dividing the order of G.*

Proof. (1) Suppose that G is a $\mathcal{T}_c$ group. Let K be any subgroup of $O_p(G)$ and X normal cyclic subgroup of K. Then X is a cyclic subnormal subgroup of G. Since, G is a $\mathcal{T}_c$-group then X is normal in G. Hence, each subgroup of the Sylow p-subgroup $O_p(G)$ of $\mathrm{Fit}(G)$ is normal cyclic sensitive in G, for all primes p dividing the order of G. Conversely, let X be a subnormal cyclic subgroup of G and X_p be a Sylow p subgroup of X, for some prime p dividing the order of X. Then X_p is contained in $O_p(G)$. By a similar argument as in Theorem 3.3.1 we get that $O_p(G)$ is a Dedekind group and normal cyclic sensitive in G. Now, X_p is a normal cyclic subgroup of $O_p(G)$. Thus, X_p is a normal subgroup of G. Since X is a direct product of its Sylow subgroups, then X is normal in G, that is G is a $\mathcal{T}_c$-group.
(2) Suppose that G is a $\mathcal{PT}_c$ group. Let K be any subgroup of $O_p(G)$ and X permutable cyclic subgroup of K. Then X is a cyclic subnormal subgroup of G. Since, G is a $\mathcal{PT}_c$-group then X is permutable in G. Hence, each subgroup of the Sylow p-subgroup $O_p(G)$ of $\mathrm{Fit}(G)$ is permutable cyclic sensitive in G, for all primes p dividing the order of G. Conversely, let X be a subnormal cyclic subgroup of G and X_p be a Sylow p subgroup of X, for some prime p dividing the order of X. Then X_p is contained in $O_p(G)$. By a similar argument as in Theorem 3.3.1 we get that $O_p(G)$ is an Iwasawa group and permutable cyclic sensitive in G. Now, X_p is permutable cyclic subgroup of $O_p(G)$. Thus, X_p is permutable subgroup of G. Since X is a direct

product of its Sylow subgroups, then X is permutable in G, that is G is a $\mathcal{PT}_c$-group.
(3) Suppose that G is a $\mathcal{PST}_c$ group. Let X be a cyclic S-permutable subgroup of $O_p(G)$. Then X is subnormal cyclic subgroup of G and G being $\mathcal{PST}_c$-group implies that X is S-permutable in G. Hence, $O_p(G)$ is S-permutable cyclic sensitive in G, for all primes p dividing the order of G. Conversely, let X be a subnormal cyclic subgroup of G and X_p be a Sylow p subgroup of X, for some prime p dividing the order of X. Then X_p is S-permutable subgroup of $O_p(G)$. Thus, X_p is S-permutable subgroup of G. Since X is a direct product of its Sylow subgroups, then X is S-permutable in G, that is G is a $\mathcal{PST}_c$-group. $\square$

Note that in part (3) of Theorems 3.3.1 and 3.3.3 we do not need every subgroup of the G_p or $O_p(G)$ to be S-permutable cyclic sensitive since every subgroup of G_p or $O_p(G)$ is S-permutable in G_p or $O_p(G)$. Also, Theorem 3.3.3 and Corollary 3.3.2 lead to the following result.

Corollary 3.3.4. *Let G be a finite group and $F := Fit(G)$.*

(1) *G is a $\mathcal{T}_c$-group if and only if F is normal cyclic sensitive and a Dedekind group.*

(2) *G is a $\mathcal{PT}_c$-group if and only if F is permutable cyclic sensitive and an Iwasawa group.*

(3) *G is a $\mathcal{PST}_c$-group if and only if F is S-permutable cyclic sensitive.*

Proof. (1) Suppose that G is a $\mathcal{T}_c$-group. Then by Lemma 1.7.2 F is a Dedekind group. Let X be a normal cyclic subgroup of F. Then X is subnormal subgroup of G and hence is normal in G, sine G is a $\mathcal{T}_c$-group. Therefore, F is normal cyclic sensitive in G. Conversely, let X be a subnormal cyclic subgroup of G. Then X is normal subgroup of F. Since F is normal cyclic sensitive in G then X is a normal subgroup of G. Hence, G is a $\mathcal{T}_c$-group.
(2) Suppose that G is a $\mathcal{PT}_c$-group. Then by Lemma 1.7.2 F is an Iwasawa group. Let X be a permutable cyclic subgroup of F. Then X is subnormal subgroup of G and hence is permutable in G, since G is $\mathcal{PT}_c$-group. Therefore, F is permutable cyclic sensitive in G. Conversely, let X be a subnormal cyclic subgroup of G. Then X is permutable subgroup of F, because F is an Iwasawa group. Since F is permutable cyclic sensitive in G then X is a permutable subgroup of G. Hence, G is a $\mathcal{PT}_c$-group.
(3) Suppose that G is a $\mathcal{PST}_c$-group. Let X be an S-permutable subgroup of F. Then X is subnormal subgroup of G and hence is S-permutable in G because G is

a $\mathcal{PST}_c$-group. Hence F is S-permutable cyclic sensitive. Conversely, let X be a subnormal cyclic subgroup of G. Then X is S-permutable cyclic subgroup of F and F is S-permutable cyclic sensitive. Hence X is S-permutable in G. That is, G is a $\mathcal{PST}_c$-group. □

3.4 A note on direct products and the intersection map

Let G_1 and G_2 be any finite groups and H_i an S-permutable sensitive in G_i. A possible question might be: *under what conditions is $H_1 \times H_2$ S-permutable sensitive in $G_1 \times G_2$?* While this question is interesting and challenging in its own right, direct products of solvable $\mathcal{PST}$-groups make it possible to answer at least part of this question. The same question applies to permutable (normal) sensitivity as well as to S-permutable (permutable or normal) *cyclic* sensitivity.

Lemma 3.4.1. *Let G_1 and G_2 be any finite groups.*

(1) *Every subgroup of $G_1 \times G_2$ is S-permutable sensitive in $G_1 \times G_2$ if and only if every subgroup of G_i is S-permutable sensitive in G_i and $(|G_i|, |\gamma_*(G_j)|) = 1$ for $i \neq j \in \{1, 2\}$.*

(2) *Every subgroup of $G_1 \times G_2$ is S-permutable cyclic sensitive in $G_1 \times G_2$ if and only if every subgroup of G_i is S-permutable cyclic sensitive in G_i.*

Proof. (1). Suppose every subgroup of $G_1 \times G_2$ is S-permutable sensitive in $G_1 \times G_2$. Then by Theorem 3.1.2 $G_1 \times G_2$ is a solvable $\mathcal{PST}$-group and hence each G_i is a solvable $\mathcal{PST}$-group, which means by the same Theorem that every subgroup of G_i is S-permutable sensitive. In addition, Theorem 2.1.2 implies that $(|G_i|, |\gamma_*(G_j)|) = 1$ for $i \neq j \in \{1, 2\}$. Conversely, if every subgroup of G_i is S-permutable sensitive then G_i is a solvable $\mathcal{PST}$-group. But, $(|G_i|, |\gamma_*(G_j)|) = 1$ for $i \neq j \in \{1, 2\}$, hence by Theorem 2.1.2 $G_1 \times G_2$ is a solvable $\mathcal{PST}$-group. Thus, by Theorem 3.1.2 every subgroup of $G_1 \times G_2$ is S-permutable sensitive in $G_1 \times G_2$.
(2). Follows by a direct application of Theorem 3.2.6 and the fact that a direct product of finitely many nilpotent groups is nilpotent. □

We sate two more results without a proof that are a direct consequence of Theorems 2.4.2, 3.2.7 and 2.6.1.

Corollary 3.4.2. *Let G_1 and G_2 be finite solvable groups. Then, every subnormal subgroup of $G_1 \times G_2$ is S-permutable cyclic sensitive if and only if every subnormal subgroup of G_i is S-permutable cyclic sensitive and $(|\mathrm{Fit}(G_i)|, |\gamma_*(G_j)|) = 1$ for $i \neq j \in \{1, 2\}$.*

Corollary 3.4.3. *Let G_1 and G_2 be finite groups so that $(|\mathrm{Fit}(G_1)|, |\mathrm{Fit}(G_2)|) = 1$. Then, every subnormal subgroup of $G_1 \times G_2$ is S-permutable (normal) cyclic sensitive in $G_1 \times G_2$ if and only if every subnormal subgroup of G_i is S-permutable (normal) cyclic sensitive in G_i.*

The above results only indicate what can lead into a rich and full of research area. The potential of this interplay between direct products and the intersection map is clear and requires further investigation.

Appendix

This section is an introduction to the local structure of the classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $\mathcal{T}_c$. By analyzing local features of these groups we might be able to answer some of the question posed in Chapters 2 and 3. What follows is an analog (local) to [12].

Definition 1. *Let G be a finite group and let p be a fixed prime number that divides the order of G.*

(i) *The group G is called a $\mathcal{PST}_{C_p}$ if every cyclic subnormal p-subgroup of G permutes with every Sylow subgroup of G.*

(ii) *The group G is called a $\mathcal{PT}_{C_p}$ if every cyclic subnormal p-subgroup of G permutes with every subgroup of G.*

(iii) *The group G is called a $\mathcal{T}_{C_p}$ if every cyclic subnormal p-subgroup of G is normal in G.*

We will say that a subgroup H of G is *subpermutable* of length 3 if H per H_1 per H_2 per G for some subgroups H_1 and H_2 of G. Similarly, we can define subpermutability of a subgroup for any length n. The reason this is emphasized is that in the next Lemma we need at most two subgroups in between to prove a claim.

Lemma 1. *Let G be any finite group and let $F_p := Syl_p(F)$.*

(i) *$G \in \mathcal{PST}_{C_p}$ if and only if a cyclic p-subgroup of G is S-per G whenever it is S-per in some S-per subgroup of G.*

(ii) *$G \in \mathcal{PT}_{C_p}$ if and only if a cyclic p-subgroup of G is permutable in G whenever it is subpermutable in G of length at most 3 and F_p is an Iwasawa group.*

(iii) *$G \in \mathcal{PT}_{C_p}$ if and only if a cyclic p-subgroup of G is normal in G whenever it is subnormal in G with defect at most 3 and F_p is a Dedekind group.*

Proof. (i). Suppose G is a $\mathcal{PST}_{C_p}$ and let H be a cyclic p-subgroup of G such that H S-per K S-per G. Then by *Kegel's Theorem* H is subnormal in K and K is subnormal in G. In particular H is subnormal subgroup of G. Since $G \in \mathcal{PST}_{C_p}$ then H S-per G. Conversely, let H be any cyclic subnormal p-subgroup of G. Then $H \leq \mathrm{Fit}(G)$. Since every Sylow subgroup of the $\mathrm{Fit}(G)$ is normal then H S-per $\mathrm{Fit}(G)$. But $\mathrm{Fit}(G)$ S-per G, hence by assumption H S-per G.

(ii). Suppose G is a $\mathcal{PT}_{C_p}$ and let H be a cyclic p-subgroup of G such that H per K per G. Then by *Ore's Theorem* H is subnormal in K and K is subnormal in G. In particular H is subnormal subgroup of G. Since $G \in \mathcal{PT}_{C_p}$ then H per G. It remains to show that F_p is an *Iwasawa* group. It is enough to show that any two cyclic subgroups of F_p permute. Let $a, b \in F_p$ such that $a, b \neq 1$. Then $\langle a \rangle$ and $\langle b \rangle$ are subnormal p-subgroups of G, and therefore are permutable in G. In particular $\langle a \rangle\langle b \rangle = \langle b \rangle\langle a \rangle$. Hence, F_p is an *Iwasawa* group.
Conversely, let H be any cyclic subnormal subgroup of G, then $H \leq F_p$ and F_p is an *Iwasawa*. Hence, H per F_p per $\mathrm{Fit}(G)$ per G. Therefore by hypothesis H is permutable in G and so, G is a $\mathcal{PT}_{C_p}$.
(iii). Suppose G is a $\mathcal{T}_{C_p}$ and let H be a cyclic p-subgroup of G such that H is subnormal in G of defect at most 3. Trivially, then $H \trianglelefteq G$. It remains to show that F_p is a *Dedekind* group. It is enough to show that any cyclic subgroups of F_p is normal. Let $a \in F_p$ such that $a \neq 1$. Then $\langle a \rangle$ is subnormal p-subgroups of G, and therefore normal in G. In particular $\langle a \rangle$ is normal subgroup of F_p. Hence, F_p is a *Dedekind* group.
Conversely, let H be any cyclic subnormal subgroup of G, then $H \leq F_p$ and F_p is a *Dedekind* group. Hence, $H \trianglelefteq F_p \trianglelefteq \mathrm{Fit}(G) \trianglelefteq G$. Therefore by hypothesis H is normal subgroup of G and so, G is a $\mathcal{T}_{C_p}$. □

For the next result we will need the following Theorem due to *Huppert*, [15] page 34.

Theorem 1. (Huppert [1961], [15]) *If G is a nonabelian finite p-group and N is a group of power automorphisms acting on G then N is a p-group.*

Lemma 2. *Let G be a finite $\mathcal{PST}_{C_p}$-group and N be a normal p-subgroup of G. Then p'-elements of G induce power automorphisms in N and $G/C_G(N)$ is nilpotent.*

Proof. Let $a \in N$ be such that $a \neq 1$ and let $Q \in Syl_Q(G)$ where $p \neq q$. Then $\langle a \rangle$ is subnormal cyclic p-subgroup of G. It follows that $\langle a \rangle Q = Q\langle a \rangle$ and $\langle a \rangle \in Syl_q(\langle a \rangle Q)$. This implies that $Q \leq N(\langle a \rangle)$. Therefore, we conclude that p'-elements of G induce power automorphisms in N.
Next, if N is not abelian then all p'-power automorphisms are trivial by Theorem 1. This implies that $G/C_G(N)$ is a p-group, and p-groups are nilpotent. If N is abelian then power automorphisms of N must be in the $Z(Aut(N))$. Since $G/C_G(N)$ is a group of power automorphisms on N then it is isomorphic to a subgroup of $Z(Aut(N))$ and hence is abelian, that is $G/C_G(N)$ is nilpotent. □

Lemma 3. *Let G be a finite solvable $\mathcal{PST}_{C_p}$-group and $L_p \in Syl_p(L)$ be normal subgroup of G. Then*

(i) *L_p is abelian.*

(ii) *G is p-supersolvable.*

(iii) *$L = L_p \times L_{p'}$, where $L_{p'}$ is a normal Hall-subgroup of L.*

Proof. Let $F_p \in Syl_p(F)$. Since F_p is subnormal subgroup of G then by *Lemma 1.2* $G/C_G(F_p)$ is nilpotent. But L is the smallest normal subgroup of G such that G/L is nilpotent. This implies that $L \leq C_G(F_p)$. Also, $L_p \leq F_p$. Therefore, $[L_p, L] \leq [F_p, L] = 1$. Since L_p is normal subgroup of L and $[L_p, L] = 1$ then $L_p \leq Z(L)$. Hence, we conclude that L_p is abelian.
To show that G is p-supersolvable we need to show that all p-chief factors of G are cyclic. First we note that G is solvable thus all p-chief factors are elementary abelian. If take any p-chief factor above L, i.e. if H/K is a p-chief factor of G such that $L \leq K \leq H \leq G$ then since G/L is nilpotent then it is supersolvable and chief factors of G/L are the chief factors of G by the *Third Isomorphism Theorem.* Hence, all chief factors of G above L are cyclic, in particular p-chief factors are cyclic. Next we note that since L_p is normal subgroup of L, then L_p has a complement, in particular L/L_p is a p'-group. Hence p-chief factors of G between L_p and L are trivial. Let H/K be a p-chief factor of G below L_p and let $P \in Syl_p(G)$. Then $PK/K \in Syl_p(G/K)$ and $H/K \cap Z(PK/K) \neq \overline{1}$. Let $\overline{x} \in H/K \cap Z(PK/K)$, then $\mathrm{ord}(\overline{x}) = p$ because H/K is elementary p-abelian. Since $O^p(G)$ acts by conjugation on H/K and PK/K normalizes $\langle \overline{x} \rangle$ then $\langle \overline{x} \rangle$ is a normal subgroup of G/K. But H/K is minimal normal subgroup, hence $H/K = \langle \overline{x} \rangle$ and $|H/K| = p$. Therefore, p-chief factors of G below L_p are all cyclic. Hence, we conclude that G is p-supersolvable.
Now, G' of a p-supersolvable group is p-nilpotent, and $L \leq G'$. Hence L is p-nilpotent which implies that $L_{p'}$ the complement of L_p is a normal Hall-subgroup of L. Hence, $L = L_p \times L_{p'}$. □

It is intuitive to suppose that if for each prime divisor p a group G is a $\mathcal{PST}_{C_p}$ then G is a $\mathcal{PST}_c$. The intersection map of subgroups provides a straightforward proof to the latter.

Corollary 1. *Let G be a finite group and $F_p \in Syl_p(Fit(G))$ for some prime divisor p of the order of G.*

(1) *G is a $\mathcal{T}_{C_p}$-group if and only if F_p is normal cyclic sensitive and a Dedekind group.*

(2) *G is a $\mathcal{PT}_{C_p}$-group if and only if F_p is permutable cyclic sensitive and an Iwasawa group.*

(3) *G is a $\mathcal{PST}_{C_p}$-group if and only if F_p is S-permutable cyclic sensitive.*

Proof. (1) Suppose that G is a $\mathcal{T}_{C_p}$-group. Then by Lemma 1 F_p is a Dedekind group. Let X be a normal cyclic subgroup of F_p. Then X is subnormal subgroup of G and hence is normal in G, since G is a $\mathcal{T}_{C_p}$-group. Therefore, F_p is normal cyclic sensitive in G. Conversely, let X be a subnormal cyclic p-subgroup of G. Then X is normal subgroup of F_p. Since F_p is normal cyclic sensitive in G then X is a normal subgroup of G. Hence, G is a $\mathcal{T}_{C_p}$-group.
(2) Suppose that G is a $\mathcal{PT}_{C_p}$-group. Then by Lemma 1 F_p is an Iwasawa group. Let X be a permutable cyclic subgroup of F_p. Then X is subnormal subgroup of G and hence is permutable in G, since G is a $\mathcal{PT}_{C_p}$-group. Therefore, F_p is permutable cyclic sensitive in G. Conversely, let X be a subnormal cyclic p-subgroup of G. Then X is permutable subgroup of F_p since F_p is an Iwasawa group by assumption. Since F_p is permutable cyclic sensitive in G then X is a permutable subgroup of G. Hence, G is a $\mathcal{PT}_{C_p}$-group.
(3) Suppose that G is a $\mathcal{PST}_{C_p}$-group. Let X be an S-permutable cyclic subgroup of F_p. Then X is subnormal subgroup of G and hence is S-permutable in G, sine G is a $\mathcal{PST}_{C_p}$-group. Therefore, F_p is S-permutable cyclic sensitive in G. Conversely, let X be a subnormal cyclic p-subgroup of G. Then X is S-permutable subgroup of F_p. Since F_p is S-permutable cyclic sensitive in G then X is an S-permutable subgroup of G. Hence, G is a $\mathcal{PST}_{C_p}$-group. □

Theorem 2. *Let G be a finite group.*

(1) *G is a $\mathcal{PST}_c$ group if and only if G is a $\mathcal{PST}_{C_p}$ for all prime divisors p of the order of G.*

(2) *G is a $\mathcal{PT}_c$ group if and only if G is a $\mathcal{PT}_{C_p}$ for all prime divisors p of the order of G.*

(3) *G is a $\mathcal{T}_c$ group if and only if G is a $\mathcal{T}_{C_p}$ for all prime divisors p of the order of G.*

Proof. (1) Suppose that G is a $\mathcal{PST}_c$-group. Then by Theorem 3.3.3 every Sylow p subgroup of $\mathrm{Fit}(G)$ is S-permutable cyclic sensitive in G, for each prime divisor p of the order of G. Hence, by Corollary 1 G is a $\mathcal{PST}_{C_p}$ for each prime p. Conversely, suppose that G is a $\mathcal{PST}_{C_p}$ for all prime divisors p of the order of G. Then each Sylow subgroup of $\mathrm{Fit}(G)$ is S-permutable cyclic sensitive by Corollary 1, and therefore by Theorem 3.3.3 G is a $\mathcal{PST}_c$.

(2) Suppose that G is a $\mathcal{PT}_c$-group. Then by Theorem 3.3.3 every subgroup of a Sylow p subgroup of $\mathrm{Fit}(G)$ is permutable cyclic sensitive in G, for each prime divisor p of the order of G. By a similar argument as in Theorem 3.3.1 we get that each Sylow subgroup of the $\mathrm{Fit}(G)$ is an Iwasawa group and permutable cyclic sensitive in G. Hence, by Corollary 1 G is a $\mathcal{PT}_{C_p}$ for each prime p. Conversely, suppose that G is a $\mathcal{PT}_{C_p}$ for all prime divisors p of the order of G. Then each Sylow subgroup of $\mathrm{Fit}(G)$ is permutable cyclic sensitive and an Iwasawa group by Corollary 1. Let F_p be a Sylow p subgroup of $\mathrm{Fit}(G)$. Let H be any subgroup of F_p and X any permutable cyclic subgroup of H. Since, F_p is an Iwasawa group then X is permutable subgroup of F_p. But F_p is also permutable cyclic sensitive in G, hence X is permutable subgroup of G. This means that every subgroup of each Sylow subgroup of the $\mathrm{FIt}(G)$ is permutable cyclic sensitive in G. Therefore, by Theorem 3.3.3 G is a $\mathcal{PT}_c$.

(3) Suppose that G is a $\mathcal{T}_c$-group. Then by Theorem 3.3.3 every subgroup of a Sylow p subgroup of $\mathrm{Fit}(G)$ is normal cyclic sensitive in G, for each prime divisor p of the order of G. By a similar argument as in Theorem 3.3.1 we get that each Sylow subgroup of the $\mathrm{Fit}(G)$ is a Dedekind group and normal cyclic sensitive in G. Hence, by Corollary 1 G is a $\mathcal{T}_{C_p}$ for each prime p. Conversely, suppose that G is a $\mathcal{T}_{C_p}$ for all prime divisors p of the order of G. Then each Sylow subgroup of $\mathrm{Fit}(G)$ is normal cyclic sensitive and a Dedekind group by Corollary 1. Let F_p be a Sylow p subgroup of $\mathrm{Fit}(G)$. Let H be any subgroup of F_p and X any normal cyclic subgroup of H. Since, F_p is a Dedekind group then X is a normal subgroup of F_p. But F_p is also normal cyclic sensitive in G, hence X is a normal subgroup of G. This means that every subgroup of each Sylow subgroup of the $\mathrm{FIt}(G)$ is normal cyclic sensitive in G. Therefore, by Theorem 3.3.3 G is a $\mathcal{T}_c$. $\square$

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Vita

Julia Chifman

Date and place of birth: April 18, 1974, Konstantinovka, Ukraine

Educational institutions attended and degrees already awarded:

B.S. , Mathematics, King's College, Wilkes-Barre, PA, 2003.

M.A., Mathematics, University of Kentucky, 2005.

Scholastic and professional honors:

Kentucky Opportunity Fellowship, 2008 - 2009.

Summer Research Support Award, 2008.

Edgar Enoch Algebra Scholarship, University of Kentucky, 2007.

Summer Research Support Award, 2006.

Graduate School Fellowship, University of Kentucky, 2003 - 2004.

The Rev. James E. Kline Award for Mathematics, King's College, 2003.

Outstanding Adult Learner of the Year, King's College, 2003.

Delta Epsilon Sigma Undergraduate Scholarship, King's College, 2002-2003.

Certificate of Honor from the President and Faculty of King's College, 2002.

Professional publications:

The intersection map of subgroups and the classes PSTc, PTc and Tc, with James C. Beidleman, Ricerche mat. (2008) 57:301-309

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